

Local Statistics of Realizable Vertex Models

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Abstract

We study planar “vertex” models, which are probability measures on edge subsets of a planar graph, satisfying certain constraints at each vertex, examples including dimer model, and 1-2 model, which we will define. We express the local statistics of a large class of vertex models on a finite hexagonal lattice as a linear combination of the local statistics of dimers on the corresponding Fisher graph, with the help of a generalized holographic algorithm. Using an $n \times n$ torus to approximate the periodic infinite graph, we give an explicit integral formula for the free energy and local statistics for configurations of the vertex model on an infinite bi-periodic graph. As an example, we simulate the 1-2 model by the technique of Glauber dynamics.

1 Introduction

A **vertex model** is a graph $G = (V, E)$ where we associate to each vertex $v \in V$ a **signature** r_v . A **local configuration** at a vertex v is a subset of incident edges of v . A **configuration** of the graph G is an edge subset of G . The signature r_v at a vertex v is a function which assigns a nonnegative real number (**weight**) to each local configuration at v . The **partition function** of the vertex model is the weighted sum of configurations $X \in \{0, 1\}^E$, where the weight of a configuration is the product of weights of local configurations, obtained by restricting the configuration at each vertex. Dimers, loop models, and random tiling models are some special examples of vertex models.

Direct computations of the partition function of a general vertex model usually require exponential time. On the other hand, using the Fisher-Kasteleyn-Temperley method [4, 5], we can efficiently count the number of perfect matchings (dimer configurations) of a finite planar graph. The idea of **generalized holographic reduction** is to reduce a vertex model on a planar graph to a dimer model on another planar graph, essentially by a linear base change, see section 3. For an original version of the holographic reduction (Valiant’s Algorithm), see [16].

However, not all satisfying assignment problems can be reduced to a perfect matching problem (“realized”) using the holographic algorithm. We study the realizability problem of the generalized algorithm for vertex models on the hexagonal

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lattice and prove that the signature of realizable models form a submanifold of positive codimension of the manifold of all signatures, see Theorem 3.6. An example of realizable models is the **1-2 model**, which is a signature on the honeycomb lattice, only one or two edges allowed to be present in each local configuration, see Figure 13, 14. The realizability problem of Valiant's algorithm is studied by Cai [2], and the realizability problem of **uniform 1-2 model** (not-all-equal relation), a special 1-2 model which assigns all the configurations weight 1, under Valiant's algorithm is studied by Schwartz and Bruck [14].

Realizable vertex models may be reduced to dimer models in more than one way, that is, using different bases. However, all the dimer models corresponding to the same vertex model are shown to be **gauge equivalent**, i.e. obtained from one another by a trivial reweighting.

One of the simplest vertex configuration models is a graph with the same signature at all vertices. Using the singular value decomposition, we prove that such models on a hexagonal lattice are realizable if and only if they are realizable under orthogonal base change. Moreover, the orthogonal realizability condition takes a very nice form; see section 3.2.

We compute the local statistics of realizable vertex models on a hexagonal lattice with the help of the generalized holographic reduction, i.e. for the natural probability measure, we compute the probabilities of given configurations at finitely many fixed vertices, which are proved to be computable by sums of finitely many Pfaffians, see Theorem 5.1 and Theorem 5.2.

The weak limit of probability measures of the vertex model on finite graphs are of considerable interest. Using an $n \times n$ torus to approximate the infinite periodic graph, we give an explicit integral formula for the probability of a specific local configuration at a fixed vertex, see section 6. These results follow from a study of the zeros of the **characteristic polynomial**, or the **spectral curve**, on the unit torus \mathbb{T}^2 . For a more general result about the intersection of the spectral curve with \mathbb{T}^2 , see [13]. For example, using our method, we compute the probability that a $\{001\}$ dimer occurs and for uniform 1-2 model, and the probability that a $\{011\}$ configuration occurs at a vertex for critical 1-2 model, see Examples 7.2 and 7.3.

The main result of this paper can be stated in the following theorems

Theorem 1.1. *For a periodic, realizable, positive-weight vertex model on a hexagonal lattice G with period 1×1 , assume the corresponding Fisher graph has positive edge weights, then the free energy of G is*

$$F := \lim_{n \rightarrow \infty} \frac{1}{n^2} \log S(G_n) = \frac{1}{8\pi^2} \iint_{|z|=1, |w|=1} \log P(z, w) \frac{dz}{iz} \frac{dw}{iw}$$

where G_n is the quotient graph $G/(n\mathbb{Z} \times n\mathbb{Z})$, $P(z, w)$ is the characteristic polynomial.

Theorem 1.2. *Assume the periodic vertex model on hexagonal lattice is realizable to the dimer model on a Fisher graph with positive, periodic edge weights, and assume the entries of the corresponding base change matrices are nonzero. Let λ_n be the probability measure defined for configurations on toroidal hexagonal lattice G_n .*

Moreover, for a configuration c at a vertices v

$$\lim_{n \rightarrow \infty} \lambda_n(c, v) = \sum_{d_j} [\prod_{i=1}^p w_{d_j}] |\text{Pf}(K_\infty^{-1})_{V(d_j)}|$$

where

$$K_\infty^{-1}((u, x_1, y_1), (v, x_2, y_2)) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} z^{x_1-x_2} w^{y_1-y_2} \frac{\text{Cof}(K(z, w))_{u,v}}{P(z, w)} \frac{dz}{iz} \frac{dw}{iw}$$

d_j are local dimer configurations on the gadget of the Fisher graph corresponding to v . $V(d_j)$ is the set of vertices involved in the configuration d_j .

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2 Background

2.1 Vertex Models

Let $\{0, 1\}^k$ denote the set of all binary sequences of length k . A **vertex model** is a graph $G = (V, E)$ where we associate to each vertex $v \in V$ a function

$$r_v : \{0, 1\}^{\deg(v)} \rightarrow \mathbb{R}^+$$

r_v is called the **signature** of the vertex model at vertex v . We give a linear ordering on the edges adjacent to v , and we fix such an ordering around each vertex once and for all. This way the binary sequences of length $\deg(v)$ are in one-to-one correspondence with the local configurations at v . Each edge corresponds to a digit; if that edge is included in the configuration, the corresponding digit is 1, otherwise the corresponding digit is 0. Hence we can also consider r_v as a column vector indexed by local configurations at v :

$$r_v = \begin{pmatrix} r_v(0...00) \\ r_v(0...01) \\ r_v(0...10) \\ \dots \\ r_v(1...11) \end{pmatrix}$$

Example 2.1 (signature of the vertex model at a vertex). Assume we have a degree-2 vertex with signature

$$r_v = \begin{pmatrix} r_v(00) \\ r_v(01) \\ r_v(10) \\ r_v(11) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$

that means we give weights to the four different local configurations as in Figure 1:



Figure 1: Relation at a Vertex

Assume G is a finite graph. We define a probability measure with sample space the set of all configurations, $\Omega = \{0, 1\}^{|E|}$. The probability of a specific configuration \mathcal{R} is

$$\lambda(\mathcal{R}) = \frac{1}{S} \prod_{v \in V} r_v(\mathcal{R}). \quad (1)$$

The product is over all vertices. $r_v(\mathcal{R})$ is the weight of the local configuration obtained by restricting \mathcal{R} to the vertex v , and S is a normalizing constant called the **partition function** for vertex models, defined to be

$$S = \sum_{\mathcal{R} \in \Omega} \prod_{v \in V} r_v(\mathcal{R}).$$

The sum is over all possible configurations of G .

Now we consider a vertex model on a \mathbb{Z}^2 -periodic planar graph G . By this we mean that G is embedded in the plane so that translations in \mathbb{Z}^2 act by signature-preserving isomorphisms of G . Examples of such graphs are the square and Fisher lattices, as shown in Figure 7. Let G_n be the quotient of G by the action of $n\mathbb{Z}^2$. It is a finite graph embedded into a torus. Let S_n be the partition function of the vertex model on G_n . The **free energy** of the infinite periodic vertex model G is defined to be

$$F := \lim_{n \rightarrow \infty} \frac{1}{n^2} \log S_n$$

2.2 Perfect Matching

For more information, see [6]. A **perfect matching**, or a **dimer cover**, of a graph is a collection of edges with the property that each vertex is incident to exactly one edge. A graph is **bipartite** if the vertices can be 2-colored, that is, colored black and white so that black vertices are adjacent only to white vertices and vice versa.

To a weighted finite graph $G = (V, E, W)$, the weight $W : E \rightarrow \mathbb{R}^+$ is a function from the set of edges to positive real numbers. We define a probability measure, called the **Boltzmann measure** μ with sample space the set of dimer covers. Namely, for a dimer cover D

$$\mu(D) = \frac{1}{Z} \prod_{e \in D} W(e)$$

where the product is over all edges present in D , and Z is a normalizing constant called the **partition function** for dimer models, defined to be

$$Z = \sum_D \prod_{e \in D} W(e),$$

the sum over all dimer configurations of G .

If we change the weight function W by multiplying the edge weights of all edges incident to a single vertex v by the same constant, the probability measure defined above does not change. So we define two weight functions W, W' to be **gauge equivalent** if one can be obtained from the other by a sequence of such multiplications.

The key objects used to obtain explicit expressions for the dimer model are **Kasteleyn matrices**. They are weighted, oriented adjacency matrices of the graph G defined as follows. A **clockwise-odd orientation** of G is an orientation of the edges such that for each face (except the infinite face) an odd number of edges point along it when traversed clockwise. For a planar graph, such an orientation always exists [5]. The Kasteleyn matrix corresponding to such a graph is a $|V(G)| \times |V(G)|$ skew-symmetric matrix K defined by

$$K_{u,v} = \begin{cases} W(uv) & \text{if } u \sim v \text{ and } u \rightarrow v \\ -W(uv) & \text{if } u \sim v \text{ and } u \leftarrow v \\ 0 & \text{else.} \end{cases}$$

It is known [4, 5, 15, 8] that for a planar graph with a clock-wise odd orientation, the partition function of dimers satisfies

$$Z = \sqrt{\det K}.$$

Now let G be a \mathbb{Z}^2 -periodic planar graph. Let G_n be a quotient graph of G , as defined before. Let $\gamma_{x,n}(\gamma_{y,n})$ be a path in the dual graph of G_n winding once around the torus horizontally (vertically). Let $E_H(E_V)$ be the set of edges crossed by $\gamma_x(\gamma_y)$. We give a **crossing orientation** for the toroidal graph G_n as follows. We orient all the edges of G_n except for those in $E_H \cup E_V$. This is possible since no other edges are crossing. Then we orient the edges of E_H as if E_V did not exist. Again this is possible since $G - E_V$ is planar. To complete the orientation, we also orient the edges of E_V as if E_H did not exist.

For $\theta, \tau \in \{0, 1\}$, let $K_n^{\theta, \tau}$ be the Kasteleyn matrix K_n in which the weights of edges in E_H are multiplied by $(-1)^\theta$, and those in E_V are multiplied by $(-1)^\tau$. It is proved in [15] that the partition function Z_n of the graph G_n is

$$Z_n = \frac{1}{2} |\text{Pf}(K_n^{00}) + \text{Pf}(K_n^{10}) + \text{Pf}(K_n^{01}) - \text{Pf}(K_n^{11})|.$$

Let $E_m = \{e_1 = u_1 v_1, \dots, e_m = u_m v_m\}$ be a subset of edges of G_n . Kenyon [7] proved that the probability of these edges occurring in a dimer configuration of G_n with respect to the Boltzmann measure P_n is

$$P_n(e_1, \dots, e_m) = \frac{\prod_{i=1}^m W(u_i v_i)}{2Z_n} |\text{Pf}(K_n^{00})_{E_m^c}^c + \text{Pf}(K_n^{10})_{E_m^c}^c + \text{Pf}(K_n^{01})_{E_m^c}^c - \text{Pf}(K_n^{11})_{E_m^c}^c|$$

where $E_m^c = V(G_n) \setminus \{u_1, v_1, \dots, u_m, v_m\}$, and $(K_n^{\theta\tau})_{E_m^c}^c$ is the submatrix of $K_n^{\theta\tau}$ whose lines and columns are indexed by E_m^c .

The asymptotic behavior of Z_n when n is large is an interesting subject. One important concept is the partition function per fundamental domain, which is defined

to be

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \log Z_n.$$

Let K_1 be a Kasteleyn matrix for the graph G_1 . Given any parameters z, w , we construct a matrix $K(z, w)$ as follows. Let $\gamma_{x,1}$, $\gamma_{y,1}$ be the paths introduced as above. Multiply $K_{u,v}$ by z if Pfaffian orientation on that edge is from u to v , otherwise multiply $K_{u,v}$ by $\frac{1}{z}$, and similarly for w on γ_y . Define the **characteristic polynomial** $P(z, w) = \det K(z, w)$. The **spectral curve** is defined to be the locus $P(z, w) = 0$.

Gauge equivalent dimer weights give the same spectral curve. That is because after Gauge transformation, the determinant multiplies by a nonzero constant, and the locus of $P(z, w)$ does not change.

A formula for enlarging the fundamental domain is proved in [3, 8]. Let $P_n(z, w)$ be the characteristic polynomial of G_n with period 1×1 , and $P_1(z, w)$ be the characteristic polynomial of G_1 , then

$$P_n(z, w) = \prod_{u^n=z} \prod_{v^n=w} P_1(u, v)$$

2.3 Matchgates, Matchgrids

A **matchgate** Γ is a planar local graph including a set X of external vertices, i.e. vertices located along the boundary of the local graph. The external vertices are ordered anti-clockwise on the boundary. Γ is called an odd(even) matchgate if it has an odd(even) number of vertices.

The **signature of the matchgate** is a vector indexed by subsets of external vertices, $\{0, 1\}^{|X|}$. For a subset $X_0 \subset X$, the entry of the signature at X_0 is the partition function of dimer configurations on a subgraph of the matchgate. The subgraph is obtained from the matchgate by removing all the external vertices in X_0 .

Example 2.2 (signature of a matchgate). *Assume we have a matchgate with external vertices 1, 2, 3, and edge weights as illustrated in the following figure:*

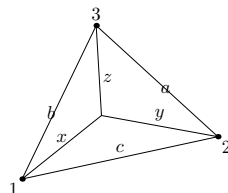


Figure 2: Matchgate

then the signature of the matchgate is

$$\mathbf{m} \begin{pmatrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{pmatrix} = \begin{pmatrix} ax + by + cz \\ 0 \\ 0 \\ x \\ 0 \\ y \\ z \\ 0 \end{pmatrix}$$

A **matchgrid** M is a weighted planar graph consisting of a collection of matchgates and connecting edges. Each connecting edge has weight 1 and joins an external vertex of a matchgate with an external vertex of another matchgate, so that every external vertex is incident to exactly one connecting edge.

3 Generalized Holographic Reduction

In this section we introduce a generalized holographic algorithm to compute the partition function of the vertex model on a finite planar graph in terms of the partition function for perfect matchings on a matchgrid. The idea is using a matchgate to replace each vertex, and perform a change of basis, such that after the base change process, the signature of a vertex becomes the signature of the corresponding matchgate. We describe the algorithm in detail as follows.

For a finite graph G , we associate to each oriented edge e a 2-dimensional vector space V_e . To the edge with the reversed orientation, the associated vector space is the dual space, i.e. $V_{-e} = V_e^*$. Give a set basis $\{f_e^0, f_e^1\}$ for each V_e , satisfying

$$f_{-e}^j(f_e^i) = \delta_{ij}.$$

Let v be a vertex with incident edges e_{l_1}, \dots, e_{l_k} , oriented away from v . The signature of the vertex model at a vertex v , r_v , can be considered as an element in $W_v = V_{e_{l_1}} \otimes \dots \otimes V_{e_{l_k}}$. Hence r_v has representations under bases $F = \{f_e^0, f_e^1\}_{e \in E}$ and $B = \{b_e^0, b_e^1\}_{e \in E}$ as follows

$$r_v = \sum_{c_{l_1}, \dots, c_{l_k}} r_v(c_{l_1} \cdots c_{l_k}) b_{e_{l_1}}^{c_{l_1}} \otimes \cdots \otimes b_{e_{l_k}}^{c_{l_k}} \quad (2)$$

$$= \sum_{c_{l_1}, \dots, c_{l_k}} r_{v,f}(c_{l_1} \cdots c_{l_k}) f_{e_{l_1}}^{c_{l_1}} \otimes \cdots \otimes f_{e_{l_k}}^{c_{l_k}} \quad (3)$$

where B are the set of standard bases for each V_e

$$b_e^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad b_e^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$c_{l_i} \in \{0, 1\}$. $c_{l_1} \cdots c_{l_k}$ are binary sequences of length k . From the definition of the signature of vertex models, obviously $r_v(c_{l_1} \cdots c_{l_k})$ is the weight of the configuration $c_{l_1} \cdots c_{l_k}$ at vertex v .

We construct a matchgrid M as follows. We replace each vertex v by a matchgate \mathcal{D}_v , such that the number of external vertices of \mathcal{D}_v is the same as the degree of v , and the edges of the vertex model graph G become connecting edges joining different matchgates in the matchgrid M . Examples of such replacements are illustrated in the following Figure.

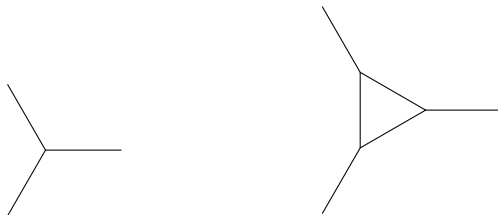


Figure 3: degree-3 vertex and matchgate



Figure 4: degree-4 vertex and matchgate

If the signature m_v of the matchgate \mathcal{D}_v satisfies

$$m_v = \sum_{c_{l_1} \cdots c_{l_k}} r_{v,f}(c_{l_1} \cdots c_{l_k}) b_{e_{l_1}}^{c_{l_1}} \otimes \cdots \otimes b_{e_{l_k}}^{c_{l_k}}, \quad (4)$$

that is, the representation of m_v under bases B is the same as the representation of r_v under bases F , then we have the following theorem:

Theorem 3.1. *Under the above base change process, the partition function of the vertex model of G is equal to the partition function of the dimer model of M .*

Proof. There is a natural mapping Φ from $\otimes_{v \in V} W_v$ to \mathbb{C} induced by $\otimes_{e \in E} \phi_e$, where ϕ_e is the natural pairing from $V_e \otimes V_e^*$ to \mathbb{C} . Note that in $\otimes_{v \in V} W_v$, each V_e and V_{-e} appear exactly once. Since the representation of m_v under bases B is the same as the representation of r_v under bases F , we have

$$\Phi(\otimes_{v \in V} m_v) = \Phi(\otimes_{v \in V} r_v). \quad (5)$$

(5) follows from the fact that each $\phi_e : V_e \otimes V_e^* \rightarrow \mathbb{C}$ is independent of bases as long as we choose the dual basis for the dual vector space. However, the left side of (5) is exactly the partition function of the dimer model of M , while the right side of (5) is exactly the partition function of the vertex model of G . \square

Define the **base change matrix at edge** e , $T_e = \begin{pmatrix} f_e^0 & f_e^1 \end{pmatrix}$. The **base change matrix at vertex** v , T_v , acting on W_v by multiplication, is defined to be

$$T_v = \bigotimes_{\{e|e \text{ is incident to } v, \text{ and oriented away from } v\}} T_e.$$

In order for a vertex model problem to be reduced to dimer model problem, one sufficient condition is that at each vertex, the signature of the vertex model r_v under the bases F is the same as the signature of the matchgate m_v under the standard bases. Namely,

$$T_v m_v = r_v. \quad (6)$$

(6) follows directly from (3) and (4).

Example 3.2. Consider the graph in Figure 3 with standard dimer signature

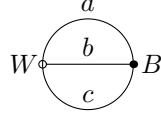


Figure 5:

$$\begin{aligned} r_w = r_b &= \begin{pmatrix} r_{000} & r_{001} & r_{010} & r_{011} & r_{100} & r_{101} & r_{110} & r_{111} \end{pmatrix}^t \\ &= \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^t \end{aligned}$$

Define the base change matrix on edges a, b, c to be

$$T_a = T_b = T_c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

By definition, $T_w = T_a \otimes T_b \otimes T_c$. Note that $T_v = (T_v^t)^{-1} = T_b$. After the base change, we have the standard loop signature

$$\begin{aligned} \tilde{r}_w = \tilde{r}_b &= T_w \cdot r_w = T_b \cdot r_b \\ &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}^t \end{aligned}$$

For instance, after the base change, the dimer configuration 001 with only c -edge occupied becomes

$$\begin{aligned} &T_w \cdot (b_0 \otimes b_0 \otimes b_1) \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \end{aligned}$$

which is the configuration 110, the loop configuration with a -edge and b -edge occupied.

However, since the number of vertices in a matchgate is either even or odd, at a vertex of degree d , the signature of a matchgate must be a 2^{d-1} dimensional subspace of \mathbb{C}^{2^d} , with those 2^{d-1} entries being 0. These entries correspond to the partition function of dimer configurations on a subgraph of the matchgate with an odd number of vertices, see example 2.1. This is the **parity constraint**. As a result, by dimension count we can see that it is not possible to use holographic algorithm to reduce all vertex models into dimer models. To characterize the special class of vertex models applicable to holographic reduction, we introduce the following definition.

Definition 3.3. A network of relations on a finite graph is **realizable**, if there exists a system of bases reducing the model to the set of perfect matchings of a matchgrid.

Definition 3.4. A network of relations is **bipartite realizable**, if it is realizable and the corresponding matchgrid is a bipartite graph.

Remark. The above generalizes Valiant's algorithm [16] in the sense that our basis can be different from edge to edge. As a consequence, our approach results in an enlargement of the dimension of realizable submanifold, which will be shown in the next section.

3.1 Realizability

We are interested in periodic vertex models on the honeycomb lattice with period $n \times n$. The quotient graph can be embedded on a torus $\mathbb{T}^2 = S^1 \times S^1$. We classify all the edges into a -type, b -type and c -type according to their direction, and assume b -type and c -type edges have the same direction as the two basic homology cycles $(1, 0)$ and $(0, 1)$ of torus, respectively, see Figure 6.

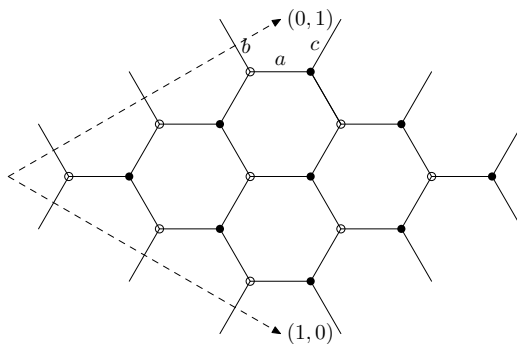


Figure 6: Periodic Honeycomb Lattice

Assume the vertex model is realizable, then each corresponding matchgate may have either an even or an odd number of vertices. By enlarging the fundamental domain, we can always assume that there are an even number of matchgates with an even number of vertices. Then by permuting rows of matrices on a finite number of edges, we can always have all the matchgates having an odd number of vertices. For example, assume we have a pair of adjacent matchgates with matrix on the

connecting edge e , $T_e = \begin{pmatrix} t_0 \\ t_1 \end{pmatrix}$ (T_e here is actually the inverse of the base change matrix defined before), as illustrated in Figure 7. Then if we assume $\tilde{T}_e = \begin{pmatrix} t_1 \\ t_0 \end{pmatrix}$

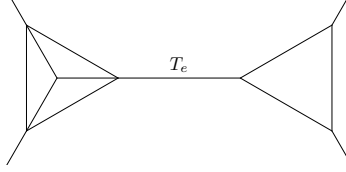


Figure 7: Permutation of Basis Vectors 1

obtained by permutating two basis vectors of T_e , we actually interchange the roles of 0 and 1 at the corresponding digit of the binary sequences as indices of signatures of the matchgate. Namely, assume $m_v = T_v r_v$, and $\tilde{m}_v = \tilde{T}_v r_v$, where $\tilde{T}_v = \tilde{T}_a \otimes \tilde{T}_b \otimes \tilde{T}_c$. Then for any binary sequence $c_1 c_2 c_3$, the entry $m_v\{c_1 c_2 c_3\}$ is the same as $\tilde{m}_v\{(1-c_1)c_2 c_3\}$, according to equation (6). If originally we have an even matchgate at v , by parity constraint, the 001, 010, 100, 111 entries of m_v are zero. After the permutation of basis vectors, \tilde{m}_v will have 101, 110, 000, 011 entries to be zero. Hence \tilde{m}_v has to be an odd matchgate, as illustrate in Figure 8. By finitely many times of such

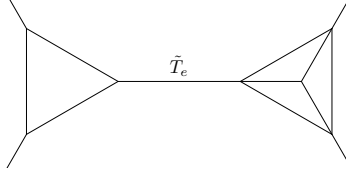


Figure 8: Permutation of Basis Vectors 2

permutations, we can move two even matchgates to adjacent positions. Then we permute the basis on the connecting edge, we can decrease the number of even matchgates by 2. Since we have an even number of even matchgates in total, if we repeat the process, in the end, all matchgates will be odd.

Assumption 3.5. *From now on, we make the following assumption:*

- All entries of base change matrices are nonzero.
- All entries of the matchgate signature are nonzero.

Since the honeycomb lattice is a bipartite graph, we can color all the vertices in black and white such that black vertices are adjacent only to white vertices and vice versa. Assume vertex signatures at black and white vertices are as follows:

$$r_w^{ij} = \begin{pmatrix} x_{000}^{ij} & x_{001}^{ij} & x_{010}^{ij} & x_{011}^{ij} & x_{100}^{ij} & x_{101}^{ij} & x_{110}^{ij} & x_{111}^{ij} \end{pmatrix}^t \quad (7)$$

$$r_b^{ij} = \begin{pmatrix} y_{000}^{ij} & y_{001}^{ij} & y_{010}^{ij} & y_{011}^{ij} & y_{100}^{ij} & y_{101}^{ij} & y_{110}^{ij} & y_{111}^{ij} \end{pmatrix}^t \quad (8)$$

where $(i, j), 1 \leq i \leq n, 1 \leq j \leq n$ is the row and column index of white and black vertices. Assume $000 = 1$, then entries of signatures are indexed from 1 to 8. Associate to an a -edge, b -edge and c -edge incident to a white or black vertices, a basis

$$\begin{aligned} T_a^{(i,j,k)} &= \begin{pmatrix} n_{01}^{(i,j,k)} & p_{01}^{(i,j,k)} \\ n_{11}^{(i,j,k)} & p_{11}^{(i,j,k)} \end{pmatrix} = \begin{pmatrix} \mathbf{n}_1^{(i,j,k)} & \mathbf{p}_1^{(i,j,k)} \end{pmatrix} \\ T_b^{(i,j,k)} &= \begin{pmatrix} n_{02}^{(i,j,k)} & p_{02}^{(i,j,k)} \\ n_{12}^{(i,j,k)} & p_{12}^{(i,j,k)} \end{pmatrix} = \begin{pmatrix} \mathbf{n}_2^{(i,j,k)} & \mathbf{p}_2^{(i,j,k)} \end{pmatrix} \\ T_c^{(i,j,k)} &= \begin{pmatrix} n_{03}^{(i,j,k)} & p_{03}^{(i,j,k)} \\ n_{13}^{(i,j,k)} & p_{13}^{(i,j,k)} \end{pmatrix} = \begin{pmatrix} \mathbf{n}_3^{(i,j,k)} & \mathbf{p}_3^{(i,j,k)} \end{pmatrix} \end{aligned}$$

where $k = 1$ for white vertices and $k = 0$ for black vertices.

By the realizability equation (6), and the fact that all matchgates are odd, the 1st, 4th, 6th, 7th entries of the matchgate signatures m_v are 0, so we have a system of 4 algebraic equations at each vertex v . For example, at each black vertex, we have

$$m_{i,j,0} = T_a^{i,j,0} \otimes T_b^{i,j,0} \otimes T_c^{i,j,0} \cdot r_b^{ij} \quad (9)$$

and the fact that the 1st entry of $m_{i,j,0}$ is 0 gives the following equation

$$\begin{aligned} &n_{01}n_{02}n_{03}y_1 + n_{01}n_{02}p_{03}y_2 + n_{01}p_{02}n_{03}y_3 + n_{01}p_{02}p_{03}y_4 \\ &+ p_{01}n_{02}n_{03}y_5 + p_{01}n_{02}p_{03}y_6 + p_{01}p_{02}n_{03}y_7 + p_{01}p_{02}p_{03}y_8 = 0 \end{aligned}$$

That the 4th, 6th, 7th entries of $m_{i,j,0}$ are 0 give three other similar equations. Similarly at each white vertex, we have

$$m_{i,j,1} = [(T_a^{i,j,1} \otimes T_b^{i,j,1} \otimes T_c^{i,j,1})^t]^{-1} \cdot r_w^{ij} \quad (10)$$

the same process gives a system of 4 equations at the white vertex.

Let $a_{lm}^{ij} = \frac{n_{lm}^{(i,j,1)}}{p_{lm}^{(i,j,1)}}$, $b_{lm}^{ij} = \frac{n_{lm}^{(i,j,0)}}{p_{lm}^{(i,j,0)}}$, $l \in \{0, 1\}, m \in \{1, 2, 3\}$. Then the equations we get are linear with respect to a_{01}^{ij} , a_{11}^{ij} , b_{01}^{ij} , b_{11}^{ij} , which can be solved explicitly.

$$a_{01}^{ij} = \frac{r^{ij} \cdot u^{ij}}{r^{ij} \cdot v^{ij}} = \frac{q^{ij} \cdot u^{ij}}{q^{ij} \cdot v^{ij}} \quad (11)$$

$$a_{11}^{ij} = \frac{s^{ij} \cdot u^{ij}}{s^{ij} \cdot v^{ij}} = \frac{p^{ij} \cdot u^{ij}}{p^{ij} \cdot v^{ij}} \quad (12)$$

$$b_{01}^{ij} = -\frac{\xi^{ij} \cdot t^{ij}}{\xi^{ij} \cdot w^{ij}} = -\frac{\kappa^{ij} \cdot t^{ij}}{\kappa^{ij} \cdot w^{ij}} \quad (13)$$

$$b_{11}^{ij} = -\frac{\sigma^{ij} \cdot t^{ij}}{\sigma^{ij} \cdot w^{ij}} = -\frac{\lambda^{ij} \cdot t^{ij}}{\lambda^{ij} \cdot w^{ij}} \quad (14)$$

where

$$\begin{cases} p^{ij} = \begin{pmatrix} a_{02}^{ij} a_{03}^{ij} & a_{02}^{ij} & a_{03}^{ij} & 1 \end{pmatrix} & q^{ij} = \begin{pmatrix} a_{02}^{ij} a_{13}^{ij} & a_{02}^{ij} & a_{13}^{ij} & 1 \end{pmatrix} \\ r^{ij} = \begin{pmatrix} a_{12}^{ij} a_{03}^{ij} & a_{12}^{ij} & a_{03}^{ij} & 1 \end{pmatrix} & s^{ij} = \begin{pmatrix} a_{12}^{ij} a_{13}^{ij} & a_{12}^{ij} & a_{13}^{ij} & 1 \end{pmatrix} \\ \xi^{ij} = \begin{pmatrix} b_{02}^{ij} b_{03}^{ij} & b_{02}^{ij} & b_{03}^{ij} & 1 \end{pmatrix} & \sigma^{ij} = \begin{pmatrix} b_{02}^{ij} b_{13}^{ij} & b_{02}^{ij} & b_{13}^{ij} & 1 \end{pmatrix} \\ \lambda^{ij} = \begin{pmatrix} b_{12}^{ij} b_{03}^{ij} & b_{12}^{ij} & b_{03}^{ij} & 1 \end{pmatrix} & \kappa^{ij} = \begin{pmatrix} b_{12}^{ij} b_{13}^{ij} & b_{12}^{ij} & b_{13}^{ij} & 1 \end{pmatrix} \\ u^{ij} = \begin{pmatrix} x_4^{ij} & -x_3^{ij} & -x_2^{ij} & x_1^{ij} \end{pmatrix} & v^{ij} = \begin{pmatrix} x_8^{ij} & -x_7^{ij} & -x_6^{ij} & x_5^{ij} \end{pmatrix} \\ w^{ij} = \begin{pmatrix} y_1^{ij} & y_2^{ij} & y_3^{ij} & y_4^{ij} \end{pmatrix} & t^{ij} = \begin{pmatrix} y_5^{ij} & y_6^{ij} & y_7^{ij} & y_8^{ij} \end{pmatrix} \end{cases} \quad (15)$$

Since $a_{0m} \neq a_{1m}$ and $b_{0m} \neq b_{1m}$, after clearing denominators and some reducing, for any (i, j) , equations (11)-(14) are equivalent to

$$2y_2y_8 - 2y_6y_4 + (b_{13} + b_{03})(y_1y_8 + y_2y_7 - y_5y_4 - y_6y_3) + 2(y_1y_7 - y_5y_3)b_{13}b_{03} = 0 \quad (16)$$

$$-2x_1x_7 + 2x_5x_3 + (a_{13} + a_{03})(x_2x_7 + x_1x_8 - x_4x_5 - x_6x_3) + 2(x_6x_4 - x_2x_8)a_{13}a_{03} = 0 \quad (17)$$

$$2y_8y_3 - 2y_4y_7 + (b_{12} + b_{02})(y_1y_8 + y_3y_6 - y_4y_5 - y_7y_2) + 2b_{12}b_{02}(y_6y_1 - y_2y_5) = 0 \quad (18)$$

$$2x_5x_2 - 2x_1x_6 + (x_1x_8 - x_2x_7 - x_5x_4 + x_3x_6)(a_{12} + a_{02}) + 2(x_7x_4 - x_3x_8)a_{02}a_{12} = 0 \quad (19)$$

If we solve $a_{02}, a_{12}, b_{02}, b_{12}$ explicitly, a similar process yields

$$2x_3x_2 - 2x_1x_4 + (x_5x_4 + x_1x_8 - x_2x_7 - x_3x_6)(a_{01} + a_{11}) + 2(x_6x_7 - x_5x_8)a_{01}a_{11} = 0 \quad (20)$$

$$2y_6y_7 - 2y_5y_8 + (y_3y_6 + y_2y_7 - y_1y_8 - y_4y_5)(b_{01} + b_{11}) + 2(y_2y_3 - y_1y_4)b_{01}b_{11} = 0 \quad (21)$$

We get two equations per edge, one involving the a-variables, the other involving the b-variables. But a-variables and b-variables are actually the same thing for each single edge. From equation (16),(17), we can solve basis entries $a_{l,3}^{i,j} = b_{l,3}^{i,j-1}$; from equation (18),(19), we can solve basis entries $a_{l,2}^{i,j} = b_{l,2}^{i-1,j}$. Finally from (11)-(14), we can solve $a_{l,1}$ and $b_{l,1}$, the only constraint left will be $a_{l,1}^{i,j} = b_{l,1}^{i,j}$, which is two polynomial equations with respect to relation signature at each a-edge. Together with Theorem 2 in appendix, we have the following theorem

Theorem 3.6. *Under assumption 3.5, the realizable signatures on the $n \times n$ periodic honeycomb lattice form a $14n^2$ dimensional submanifold of the $16n^2$ dimensional manifold of all positive signatures; the bipartite realizable signatures on the $n \times n$ periodic honeycomb lattice form a $12n^2$ dimensional submanifold of the $16n^2$ dimensional manifold of all positive signatures.*

For the exact realizability condition, see the appendix.

Under assumption 3.5, the weight $\{111\}$ is non-vanishing at each matchgate. Since the probability measure will not change if all entries of the signature at one vertex are multiplied by a constant, we can assume at each matchgate, the weight of $\{111\}$ is 1. Therefore, we have

Theorem 3.7. *A realizable vertex model on a finite hexagonal lattice G can always be transformed to dimers on M , a Fisher graph as shown in Figure 9, with the partition function of the vertex model of G equal to the partition function of the dimer model of M , up to multiplication of a constant.*

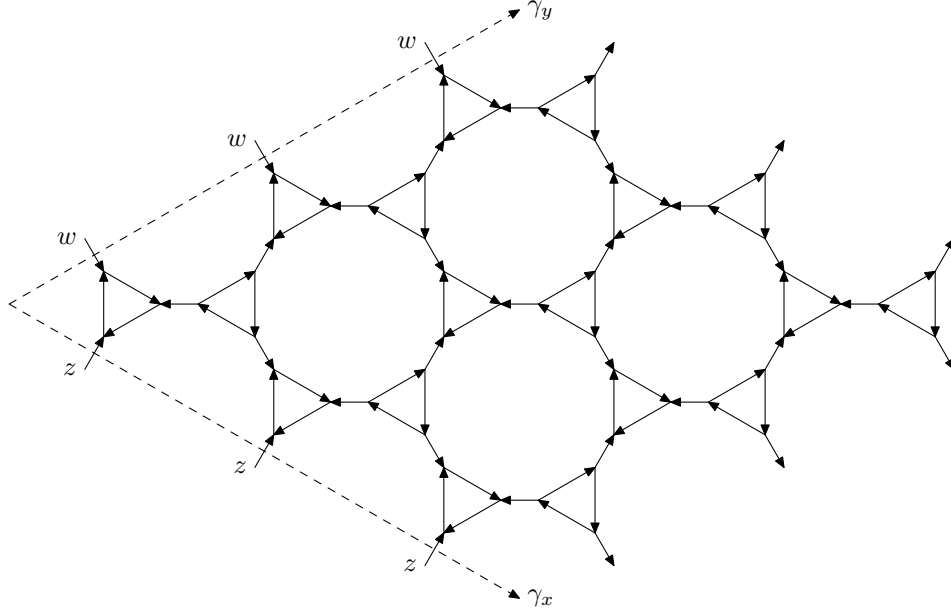


Figure 9: Matchgrid with 3×3 Fundamental Domain

It is possible to construct a matchgrid with different weights to produce the same vertex model. Since the holographic reduction is an invertible process, by which we mean that because the base change matrices are nonsingular, we can always achieve the matchgate signature from the vertex signature and vice versa, there is an equivalence relation on dimer models producing the same vertex model.

Definition 3.8. *A vertex model is holographically equivalent to a dimer model on a matchgrid, if it can be reduced to the dimer model on the matchgrid using the holographic algorithm, in such a way that partition function of the vertex model corresponds to perfect matchings of the matchgrid. Two matchgrids are holographically equivalent, if they are holographically equivalent to the same vertex model.*

Proposition 3.9. *Holographically equivalent matchgrids give rise to gauge equivalent dimer models. Therefore, they have the same probability measure.*

Proof. See the appendix. □

3.2 Orthogonal Realizability

Definition 3.10. *A vertex model is orthogonally realizable if it is realizable by an orthonormal base change matrix on each edge.*

Consider a single vertex. To the incident edges of the vertex, associate matrices U_1, U_2, U_3 . Without loss of generality, we can assume $U_1, U_2, U_3 \in SO(2)$. In fact, if $\det U_i = -1$, we multiply the first row of U_i by -1 . The new signature of the matchgate will be multiplied by -1 . This change does not violate the parity constraint.

Assume $U_i = \begin{pmatrix} \cos \alpha_i & \sin \alpha_i \\ -\sin \alpha_i & \cos \alpha_i \end{pmatrix}$. Assume $U = \otimes_{i=1}^3 U_i$, then $U \in SO(8)$, each term of which is product of three of $\cos \alpha_i, \sin \alpha_i$. Moreover, the eigenvalues are $e^{(\pm\alpha_1 \pm \alpha_2 \pm \alpha_3)\sqrt{-1}}$. Using trigonometric identities, each entry is a linear combination of $\cos(\pm\alpha_1 \pm \alpha_2 \pm \alpha_3)$ and $\sin(\pm\alpha_1 \pm \alpha_2 \pm \alpha_3)$. If we further define

$$\begin{aligned} g &= (g_1 \ g_2 \ g_3 \ g_4 \ g_5 \ g_6 \ g_7 \ g_8)' \\ h &= (0 \ h_2 \ h_3 \ 0 \ h_5 \ 0 \ 0 \ h_8)' \\ \gamma &= \alpha_1 + \alpha_2 - \alpha_3; \ \psi = \alpha_1 + \alpha_3 - \alpha_2; \ \varphi = \alpha_2 + \alpha_3 - \alpha_1 \\ j_1 &= g_4 + g_6 + g_7 - g_1; \ j_2 = g_1 + g_6 + g_7 - g_4; \\ j_3 &= g_1 + g_4 + g_7 - g_6; \ j_4 = g_1 + g_4 + g_6 - g_7; \\ j_5 &= g_3 + g_5 + g_8 - g_2; \ j_6 = g_2 + g_5 + g_8 - g_3; \\ j_7 &= g_2 + g_3 + g_8 - g_5; \ j_8 = g_2 + g_3 + g_5 - g_8 \end{aligned}$$

$$\begin{aligned} P &= j_2 \cos(\varphi) + j_7 \sin(\varphi); \ Q = j_3 \cos(\psi) + j_6 \sin(\psi); \ R = j_4 \cos(\gamma) + j_5 \sin(\gamma) \\ K &= j_1 \cos(\varphi + \psi + \gamma) - j_8 \sin(\varphi + \psi + \gamma) \\ H &= j_7 \cos(\varphi) - j_2 \sin(\varphi); \ L = j_6 \cos(\psi) - j_3 \sin(\psi); \ M = j_5 \cos(\gamma) - j_4 \sin(\gamma) \\ N &= j_8 \cos(\varphi + \psi + \gamma) + j_1 \sin(\varphi + \psi + \gamma) \end{aligned}$$

then if $Ug = h$, we have

$$0 = P = Q = R = K \tag{22}$$

$$4h_2 = H + L - M + N \tag{23}$$

$$4h_3 = H - L + M + N \tag{24}$$

$$4h_5 = N + M + L - H \tag{25}$$

$$4h_8 = H + L + M - N \tag{26}$$

By (22), we have $\tan \varphi = -\frac{j_2}{j_7}, \tan \psi = -\frac{j_3}{j_6}, \tan \gamma = -\frac{j_4}{j_5}, \tan(\varphi + \psi + \gamma) = \frac{j_1}{j_8}$. If we define

$$\begin{aligned} t(u, v) &= \frac{u + v}{1 - uv} \\ tt(a, b, c, d) &= \frac{a + b + c + d - abc - abd - acd - bcd}{1 - ab - ac - ad - bc - bd - cd + abcd} \end{aligned}$$

Then the following theorem holds:

Theorem 3.11. *A vertex model on a periodic honeycomb lattice with period $n \times n$ is orthogonally realizable if and only if its signatures satisfy the following system of*

equations

$$\begin{aligned}
& tt\left(-\frac{z_7^{ijk}}{z_2^{ijk}}, -\frac{z_6^{ijk}}{z_3^{ijk}}, -\frac{z_4^{ijk}}{z_5^{ijk}}, -\frac{z_1^{ijk}}{z_8^{ijk}}\right) = 0 \quad \forall i, j, k \\
& \left. \begin{aligned}
& t\left(-\frac{z_6^{ij0}}{z_3^{ij0}}, -\frac{z_4^{ij0}}{z_5^{ij0}}\right) = t\left(-\frac{z_6^{ij1}}{z_3^{ij1}}, -\frac{z_4^{ij1}}{z_5^{ij1}}\right) \\
& t\left(-\frac{z_3^{ij-1,0}}{z_6^{ij-1,0}}, -\frac{z_2^{ij-1,0}}{z_7^{ij-1,0}}\right) = t\left(-\frac{z_3^{ij1}}{z_6^{ij1}}, -\frac{z_2^{ij1}}{z_7^{ij1}}\right) \\
& t\left(-\frac{z_7^{i-1,j0}}{z_2^{i-1,j0}}, -\frac{z_4^{i-1,j0}}{z_5^{i-1,j0}}\right) = t\left(-\frac{z_7^{ij1}}{z_2^{ij1}}, -\frac{z_4^{ij1}}{z_5^{ij1}}\right)
\end{aligned} \right\} \rightarrow \forall i, j
\end{aligned}$$

where $z_1^{ij1} = x_4 + x_6 + x_7 - x_1$, $z_2^{ij1} = x_3 + x_5 + x_8 - x_2$, $z_3^{ij1} = x_2 + x_5 + x_8 - x_3$, $z_4^{ij1} = x_1 + x_6 + x_7 - x_4$, $z_5^{ij1} = x_2 + x_3 + x_8 - x_5$, $z_6^{ij1} = x_1 + x_4 + x_7 - x_6$, $z_7^{ij1} = x_1 + x_4 + x_6 - x_7$, $z_8^{ij1} = x_2 + x_3 + x_5 - x_8$, and the same relation for z_l^{ij0} and y_1, \dots, y_8 . x 's and y 's are defined by (7) and (8).

It is trivial to verify these equations in any given situation.

Definition 3.12. A vertex model on a periodic honeycomb lattice with period $n \times n$ is positively orthogonally realizable if it is orthogonal realizable and for each vertex (i, j, k) , there exists angles $\varphi^{ijk}, \psi^{ijk}, \gamma^{ijk}$, such that

$$\begin{aligned}
\sin \varphi &= \frac{z_4^{ijk}}{\sqrt{(z_4^{ijk})^2 + (z_5^{ijk})^2}}; \\
\sin \psi &= \frac{z_6^{ijk}}{\sqrt{(z_3^{ijk})^2 + (z_6^{ijk})^2}}; \\
\sin \gamma &= \frac{z_7^{ijk}}{\sqrt{(z_7^{ijk})^2 + (z_2^{ijk})^2}} \\
\sin(-\gamma - \varphi - \psi) &= \frac{z_1^{ijk}}{\sqrt{(z_1^{ijk})^2 + (z_8^{ijk})^2}}
\end{aligned}$$

Proposition 3.13. If a vertex model on a bi-periodic hexagonal lattice is orthogonally realizable, then the corresponding dimer configuration has positive edge weights.

Proof. Under the assumption that the vertex model have nonnegative signature, we have

$$\begin{aligned}
z_1 &\leq z_4 + z_6 + z_7; \quad z_4 \leq z_1 + z_6 + z_7; \quad z_6 \leq z_1 + z_4 + z_7; \quad z_7 \leq z_4 + z_6 + z_7 \\
z_2 &\leq z_3 + z_5 + z_8; \quad z_3 \leq z_2 + z_5 + z_8; \quad z_5 \leq z_2 + z_3 + z_8; \quad z_8 \leq z_2 + z_3 + z_5
\end{aligned}$$

at all vertices. Since at least three of z_1, z_4, z_6, z_7 are nonnegative, similarly for z_2, z_3, z_5, z_8 , if we take absolute value for all z_i , the above inequalities also hold.

therefore

$$\begin{aligned}
& \sqrt{z_4^2 + z_5^2} + \sqrt{z_6^2 + z_3^2} + \sqrt{z_2^2 + z_7^2} - \sqrt{z_1^2 + z_8^2} \geq 0 \\
& \sqrt{z_4^2 + z_5^2} + \sqrt{z_6^2 + z_3^2} + \sqrt{z_1^2 + z_8^2} - \sqrt{z_2^2 + z_7^2} \geq 0 \\
& \sqrt{z_4^2 + z_5^2} + \sqrt{z_1^2 + z_8^2} + \sqrt{z_2^2 + z_7^2} - \sqrt{z_3^2 + z_6^2} \geq 0 \\
& \sqrt{z_1^2 + z_8^2} + \sqrt{z_6^2 + z_3^2} + \sqrt{z_2^2 + z_7^2} - \sqrt{z_4^2 + z_5^2} \geq 0
\end{aligned}$$

Under the assumption that both the vertex model are positively orthogonal realizable, the left side of the above inequalities are exactly edge weights of corresponding matchgates. \square

Theorem 3.14. *If all matchgates have the same signature, for a generic choice of signature, a vertex model on a hexagonal lattice is realizable if and only if it is orthogonal realizable.*

Proof. Obviously orthogonal realizability implies realizability, we only need to show that realizability implies orthogonal realizability.

Assume $r = (x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ x_8)$ is a realizable signature for the vertex model. By lemma 4.1, we can assume the corresponding bases on all edges have real entries. Consider the singular value decomposition for the base change matrix on each edge. Without loss of generality, assume

$$T_i = U_i \begin{pmatrix} 1 & 0 \\ 0 & \lambda_i \end{pmatrix} V_i \quad i = 1, 2, 3$$

where $U_i, V_i \in O(2)$, and λ_i is a nonnegative real number. Assume

$$v = (\otimes_{i=1}^3 V_i) r^t = (v_1 \ v_2 \ v_3 \ v_4 \ v_5 \ v_6 \ v_7 \ v_8)^t$$

Then we can consider $(\otimes_{i=1}^3 \begin{pmatrix} 1 & 0 \\ 0 & \lambda_i \end{pmatrix})v$ and $(\otimes_{i=1}^3 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix})v$ to be signatures of black and white vertices that are orthogonal realizable. Then by the conditions of orthogonal realizability, we have

$$\frac{v_1 + \lambda_1 \lambda_3 v_6 + \lambda_1 \lambda_2 v_7 - \lambda_2 \lambda_3 v_4}{\lambda_3 v_2 + \lambda_2 v_3 + \lambda_1 \lambda_2 \lambda_3 v_8 - \lambda_1 v_5} = \frac{v_1 + \frac{1}{\lambda_1 \lambda_3} v_6 + \frac{1}{\lambda_1 \lambda_2} v_7 - \frac{1}{\lambda_2 \lambda_3} v_4}{\frac{1}{\lambda_3} v_2 + \frac{1}{\lambda_2} v_3 + \frac{1}{\lambda_1 \lambda_2 \lambda_3} v_8 - \frac{1}{\lambda_1} v_5} \quad (27)$$

$$\frac{v_1 + \lambda_1 \lambda_3 v_6 - \lambda_1 \lambda_2 v_7 + \lambda_2 \lambda_3 v_4}{-\lambda_3 v_2 + \lambda_2 v_3 + \lambda_1 \lambda_2 \lambda_3 v_8 + \lambda_1 v_5} = \frac{v_1 + \frac{1}{\lambda_1 \lambda_3} v_6 - \frac{1}{\lambda_1 \lambda_2} v_7 + \frac{1}{\lambda_2 \lambda_3} v_4}{-\frac{1}{\lambda_3} v_2 + \frac{1}{\lambda_2} v_3 + \frac{1}{\lambda_1 \lambda_2 \lambda_3} v_8 + \frac{1}{\lambda_1} v_5} \quad (28)$$

$$\frac{v_1 - \lambda_1 \lambda_3 v_6 + \lambda_1 \lambda_2 v_7 + \lambda_2 \lambda_3 v_4}{\lambda_3 v_2 - \lambda_2 v_3 + \lambda_1 \lambda_2 \lambda_3 v_8 + \lambda_1 v_5} = \frac{v_1 - \frac{1}{\lambda_1 \lambda_3} v_6 + \frac{1}{\lambda_1 \lambda_2} v_7 + \frac{1}{\lambda_2 \lambda_3} v_4}{\frac{1}{\lambda_3} v_2 - \frac{1}{\lambda_2} v_3 + \frac{1}{\lambda_1 \lambda_2 \lambda_3} v_8 + \frac{1}{\lambda_1} v_5} \quad (29)$$

$$\frac{-v_1 + \lambda_1 \lambda_3 v_6 + \lambda_1 \lambda_2 v_7 + \lambda_2 \lambda_3 v_4}{\lambda_3 v_2 + \lambda_2 v_3 - \lambda_1 \lambda_2 \lambda_3 v_8 + \lambda_1 v_5} = \frac{-v_1 + \frac{1}{\lambda_1 \lambda_3} v_6 + \frac{1}{\lambda_1 \lambda_2} v_7 + \frac{1}{\lambda_2 \lambda_3} v_4}{\frac{1}{\lambda_3} v_2 + \frac{1}{\lambda_2} v_3 - \frac{1}{\lambda_1 \lambda_2 \lambda_3} v_8 + \frac{1}{\lambda_1} v_5} \quad (30)$$

If we transform fractions into polynomials, and consider (27)+(28)–(29)–(30), (27)+(29)–(28)–(30), and (27)+(30)–(28)–(29), then

$$\begin{aligned}(\lambda_1 + 1)(\lambda_1 - 1)(v_4v_8 + v_1v_5 + v_3v_7 + v_2v_6) &= 0 \\(\lambda_2 + 1)(\lambda_2 - 1)(v_3v_4 + v_5v_6 + v_1v_2 + v_7v_8) &= 0 \\(\lambda_3 + 1)(\lambda_3 - 1)(v_4v_2 + v_7v_5 + v_3v_1 + v_8v_6) &= 0\end{aligned}$$

Then for a generic choice of signature, we must have $\lambda_1 = \lambda_2 = \lambda_3 = 1$. \square

4 Characteristic Polynomial

Assume all entries of the vertex signatures are strictly positive. In this section we prove some interesting properties of the characteristic polynomial.

Lemma 4.1. *For realizable vertex model with positive signature and period 1×1 , there exists a realization of base change over $\mathbf{GL}_2(\mathbb{R})$, (i. e. one can take base change matrices to be real), with the property that, at each edge, $\frac{n_0 n_1}{p_0 p_1} < 0$.*

Proof. For 1×1 fundamental domain, $a_{lk} = b_{lk}$, $\forall l, k$. By (11)–(14), we have

$$\begin{aligned}a_{01} &= -\frac{p \cdot t}{p \cdot w} = -\frac{s \cdot t}{s \cdot w} = \frac{r \cdot u}{r \cdot v} = \frac{q \cdot u}{q \cdot v} \\a_{11} &= -\frac{q \cdot t}{q \cdot w} = -\frac{r \cdot t}{r \cdot w} = \frac{s \cdot u}{s \cdot v} = \frac{p \cdot u}{p \cdot v},\end{aligned}$$

From which we derive

$$(p \cdot t)(r \cdot v) + (p \cdot w)(r \cdot u) - (r \cdot t)(p \cdot v) - (r \cdot w)(p \cdot u) = 0 \quad (31)$$

$$(s \cdot t)(q \cdot v) + (s \cdot w)(q \cdot u) - (q \cdot t)(s \cdot v) - (q \cdot w)(s \cdot u) = 0 \quad (32)$$

$$(p \cdot t)(q \cdot v) + (p \cdot w)(q \cdot u) - (q \cdot t)(p \cdot v) - (q \cdot w)(p \cdot u) = 0 \quad (33)$$

$$(s \cdot t)(r \cdot v) + (s \cdot w)(r \cdot u) - (r \cdot t)(s \cdot v) - (r \cdot w)(s \cdot u) = 0. \quad (34)$$

Since each base change matrix is invertible, we have $a_{0i} \neq a_{1i}$. Plugging (15) into (31)–(34), and factor out $(a_{02} - a_{12})$, or $(a_{03} - a_{13})$, we obtain that a_{03} , a_{13} are two roots of quadratic polynomial

$$\begin{aligned}(x_6y_5 + y_7x_8 + y_1x_2 + y_3x_4)z^2 + (-y_7x_7 - y_3x_3 + y_2x_2 + y_4x_4 + x_6y_6 - x_5y_5 + y_8x_8 - y_1x_1)z \\ - y_6x_5 - x_3y_4 - y_2x_1 - x_7y_8 = 0,\end{aligned}$$

and a_{02} , a_{12} are two roots of quadratic polynomial

$$\begin{aligned}(x_3y_1 + y_6x_8 + y_2x_4 + y_5x_7)z^2 + (y_8x_8 - x_6y_6 - x_5y_5 + y_7x_7 - y_2x_2 + y_3x_3 + y_4x_4 - y_1x_1)z \\ - y_8x_6 - y_7x_5 - x_1y_3 - y_4x_2 = 0\end{aligned}$$

Under the assumption that x_i, y_j are positive, these polynomials have real roots, since the products of two roots are always negative. Then the lemma follows. \square

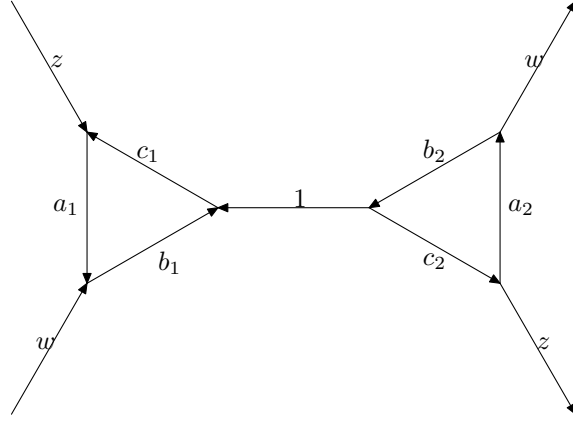


Figure 10: Weighted 1×1 Fundamental Domain

By definition, the characteristic polynomial $P(z, w)$ is the determinant of an 6×6 matrix $K(z, w)$ whose rows and columns are indexed by the 6 vertices in a 1×1 fundamental domain, see Figure 8:

$$P(z, w) = \det \begin{pmatrix} 0 & c_1 & -b_1 & -1 & 0 & 0 \\ -c_1 & 0 & a_1 & 0 & -\frac{1}{z} & 0 \\ b_1 & -a_1 & 0 & 0 & 0 & -\frac{1}{w} \\ 1 & 0 & 0 & 0 & c_2 & -b_2 \\ 0 & z & 0 & -c_2 & 0 & a_2 \\ 0 & 0 & w & b_2 & -a_2 & 0 \end{pmatrix}$$

$$= (z + \frac{1}{z})(ab - c) + (w + \frac{1}{w})(ac - b) + (\frac{z}{w} + \frac{w}{z})(bc - a) + a^2 + b^2 + 1 + c^2$$

where

$$a = a_1 a_2 \quad b = b_1 b_2 \quad c = c_1 c_2 \quad (35)$$

and we consider the non-degenerate case, which means $ac - b \neq 0$, $bc - a \neq 0$, $ab - c \neq 0$. We have the following lemma:

Lemma 4.2. *For realizable vertex model with period 1×1 , let a, b, c, d be the product of dimer weights (35) after holographic reduction, under assumption 3.4, we have the following inequalities:*

$$(a + b - c - d)(a + c - b - d)(a + d - b - c) > 0$$

$$a + b + c + d > 0.$$

Proof. For generic choice of vertex signature, we can assume quotients of basis entries $a_{ij} = \frac{n_{ij}}{p_{ij}}$ are finite. Apply the realizability equation (6) to a 1×1 quotient graph, we have 8 equations for each black vertex, and 8 equations for each white vertex. At each black vertex, 4 equations has 0 on the right side, and 4 equations with a_1, b_1, c_1, d_1 on the right side. We divide the 8 equations into 4 groups, each of which has 1 equation

with 0 on the right, 1 equation with a non-vanishing edge weight on the right. We take the difference of the two equations in each group, and we get 4 new equations with a non-vanishing edge weight on the right. We perform the same procedure for each white vertex. Then we multiply and add those equations correspondingly, we obtain

$$\begin{aligned} a + d - b - c &= (-a_{03} + a_{13})(y_3x_4 + x_8y_7 + x_6y_5 + y_1x_2) \\ a + c - b - d &= (a_{02} - a_{12})(y_1x_3 + y_2x_4 + y_6x_8 + y_5x_7) \\ a + b - c - d &= (a_{01} - a_{11})(y_1x_5 + y_4x_8 + y_2x_6 + y_3x_7) \end{aligned}$$

Moreover, since holographic reduction leaves the partition function invariant,

$$a + b + c + d = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 + x_6y_6 + x_7y_7 + x_8y_8$$

where the left side is the partition function of dimers of the 1×1 quotient graph, and the right side is the partition function of the vertex model, see Figure 10. According to lemma 5.1, a_{0k} and a_{1k} are two real numbers with opposite sign. Without loss of generality, we can assume a_{02} and a_{03} are both positive. At the black vertices, the $\{000\}$ entry of matchgate signature corresponds to partition functions of perfect matchings of the generator with all output vertices kept. Since there are odd number of vertices, no perfect matching exists, therefore

$$a_{01} = -\frac{a_{02}a_{03}y_5 + a_{02}y_6 + a_{03}y_7 + y_8}{a_{02}a_{03}y_1 + a_{02}y_2 + a_{03}y_3 + y_4} < 0$$

we have

$$-(a_{01} - a_{11})(a_{02} - a_{12})(a_{03} - a_{13}) > 0$$

The lemma follows from the assumption that entries of relation signatures are positive. \square

The expression of the characteristic polynomial shows that $P(z, w)$ is a smooth function on $\mathbb{T}^2 = \{(z, w) | |z| = 1, |w| = 1\}$. We are interested in the intersection of spectral curve $P(z, w) = 0$ with unit torus \mathbb{T}^2 , because it has implications on the convergence rate of correlations. Theorem 4.4 describes the behavior of $P(z, w)$ on \mathbb{T}^2 . Before stating the theorem, we mention an elementary lemma:

Lemma 4.3. *Let $f(\phi) = A \sin \phi + B \cos \phi + C$, where A, B, C are real. If $C \geq 0$ and $A^2 + B^2 - C^2 \leq 0$, then $f(\phi) \geq 0$ for any $\phi \in \mathbb{R}$.*

Theorem 4.4. *Under assumption 3.4, either the spectral curve after holographic reduction is disjoint from \mathbb{T}^2 , or intersects \mathbb{T}^2 at a single real node, that is, for some $(z_0, w_0) = (\pm 1, \pm 1)$,*

$$P(z, w) = \alpha(z - z_0)^2 + \beta(z - z_0)(w - w_0) + \gamma(w - w_0)^2 + \dots,$$

where $\beta^2 - 4\alpha\gamma \leq 0$.

Proof. Let

$$\begin{aligned} Q(\theta, \phi) &= P(e^{i\theta}, w^{i\phi}) \\ &= [2(ac - b) + 2(bc - a) \cos \theta] \cos \phi + 2(bc - a) \sin \theta \sin \phi \\ &\quad + 2(ab - c) \cos \theta + a^2 + b^2 + c^2 + 1. \end{aligned}$$

Consider $Q(\theta, \phi)$ as a trigonometric polynomial with respect to ϕ ,

$$Q(\theta, \phi) = A \sin \phi + B \cos \phi + C$$

where

$$\begin{aligned} A &= 2(ac - b) + 2(bc - a) \cos \theta \\ B &= 2(bc - a) \sin \theta \\ C &= a^2 + b^2 + c^2 + 1 + 2(ab - c) \cos \theta \geq 0. \end{aligned}$$

Define

$$\begin{aligned} g(\cos \theta) &= C^2 - A^2 - B^2 \\ &= 4(ab - c)^2 \cos^2 \theta + 4(ab + c)(a^2 + b^2 - c^2 - 1) \cos \theta \\ &\quad - 4(bc - a)^2 - 4(ac - b)^2 + (a^2 + b^2 + c^2 + 1)^2, \end{aligned}$$

then g is a quadratic polynomial with respect to $\cos \theta$ with discriminant

$$\Delta = 64abc(a + b - c - 1)(a + b + c + 1)(a + 1 - b - c)(a + c - b - 1).$$

The minimal value of $g(t)$ is attained at

$$t_0 = -\frac{(ab + c)(a^2 + b^2 - c^2 - 1)}{2(ab - c)^2},$$

where

$$g(t_0) = -\frac{\Delta}{16(ab - c)^2}$$

If $abc > 0$,

$$g(\cos \theta) = [2(ab + c) \cos \theta + a^2 + b^2 - c^2 - 1]^2 + 16abc \sin^2 \theta \geq 0$$

therefore $Q(\theta, \phi) \geq 0$, and $Q(\theta, \phi) = 0$ only if $\sin \theta = \sin \phi = 0$.

If $abc < 0$, lemma 4.2 implies that $\Delta < 0$, then $g(\cos \theta) > 0$, therefore $Q(\theta, \phi) > 0$.

If $abc = 0$, we have $\left| \frac{ab+c}{ab-c} \right| = 1$. Moreover,

$$(a^2 + b^2 - c^2 - 1)^2 - 4(ab - c)^2 = (a + 1 + b + c)(a - 1 - b + c)(a + 1 - b - c)(a - 1 + b - c) > 0,$$

which implies

$$|t_0| = \left| \frac{ab + c}{ab - c} \right| \left| \frac{a^2 + b^2 - c^2 - 1}{2(ab - c)} \right| > 1$$

then $g(\cos \theta) > 0$, and $Q(\theta, \phi) > 0$.

Therefore the only possible intersection of the spectral curve with \mathbb{T}^2 is real point. It is trivial to check that $\frac{\partial P}{\partial z}|_{(\pm 1, \pm 1)} = 0$, $\frac{\partial P}{\partial w}|_{(\pm 1, \pm 1)} = 0$, and $Hessian[P(z, w)]$ is positive definite wherever a real intersection of $P(z, w) = 0$ and \mathbb{T}^2 exists, then the theorem follows. \square

5 Local Statistics

5.1 Configuration at Single Vertex

We are interested in the probability of a specified configuration at a single vertex. Consider a realizable vertex model on a planar finite hexagonal lattice, whose partition function is S . Assume both the vertex with specified signature and its neighbors lie in the interior of the graph. We work out an explicit example here; other cases are very similar. If only the configuration $\{011\}$ is allowed at a black vertex v , we get a new vertex model, assume partition function is S_{001} . Then the probability that the local configuration $\{001\}$ appear at v is

$$Pr(\{001\}, v) = \frac{S_{001}}{S}.$$

Hence the local statistics problem reduces to the problem of how to compute S_{001} efficiently. We will show that S_{001} can also be computed by the dimer technique with the help of the holographic algorithm.

Let us denote the adjacent vertices of v by v_a, v_b, v_c , according to the direction of the connecting edges, as illustrated in Figure 11.

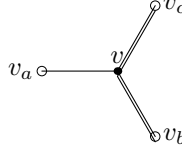


Figure 11: Configuration 011 at a Black Vertex

Let r_v, r_a, r_b, r_c denote the signatures of relations at vertices v, v_a, v_b, v_c , then we have

$$\begin{pmatrix} r_v\{000\} \\ r_v\{001\} \\ r_v\{010\} \\ r_v\{011\} \\ r_v\{100\} \\ r_v\{101\} \\ r_v\{110\} \\ r_v\{111\} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} r_a\{000\} \\ r_a\{001\} \\ r_a\{010\} \\ r_a\{011\} \\ r_a\{100\} \\ r_a\{101\} \\ r_a\{110\} \\ r_a\{111\} \end{pmatrix} = \begin{pmatrix} x_1^a \\ x_2^a \\ x_3^a \\ x_4^a \\ \star \\ \star \\ \star \\ \star \end{pmatrix}$$

$$\begin{pmatrix} r_b\{000\} \\ r_b\{001\} \\ r_b\{010\} \\ r_b\{011\} \\ r_b\{100\} \\ r_b\{101\} \\ r_b\{110\} \\ r_b\{111\} \end{pmatrix} = \begin{pmatrix} \star \\ \star \\ x_3^b \\ x_4^b \\ \star \\ \star \\ x_7^b \\ x_8^b \end{pmatrix} \quad \begin{pmatrix} r_c\{000\} \\ r_c\{001\} \\ r_c\{010\} \\ r_c\{011\} \\ r_c\{100\} \\ r_c\{101\} \\ r_c\{110\} \\ r_c\{111\} \end{pmatrix} = \begin{pmatrix} \star \\ x_2^c \\ \star \\ x_4^c \\ \star \\ x_6^c \\ \star \\ x_8^c \end{pmatrix}$$

Here \star means the entry at the position can be arbitrary. This is because we only allow the configuration $\{011\}$ at v , therefore the only configurations which actually affect the partition function of the vertex model will be those who do not occupy the a-edge of v_a , and occupy both the b-edge for v_b and the c-edge for v_c . We will split each of r_a, r_b, r_c into 2 parts, namely, $r_l = r_l(0) + r_l(1)$, $l = a, b, c$, below. By definition, the partition function of the vertex model with signature r, r_a, r_b, r_c is a sum of 8 terms, each of which is the partition function of a vertex model with signature $r_v, r_a(i), r_b(j), r_c(k)$ $i, j, k \in \{0, 1\}$. That is,

$$S_{001} = S_{\{r_v, r_a, r_b, r_c\}} = \sum_{i, j, k \in \{0, 1\}} S_{\{r_v, r_a(i), r_b(j), r_c(k)\}} \quad (36)$$

For each $S_{\{r_v, r_a(i), r_b(j), r_c(k)\}}$, we give new bases on edges adjacent to v , such that $S_{\{r_v, r_a(i), r_b(j), r_c(k)\}}$ become the partition function of certain local dimer configurations. Namely

$$S_{\{r_v, r_a(i), r_b(j), r_c(k)\}} \stackrel{\text{base change}}{=} Z_{\{m_v(i, j, k), m_a(i), m_b(j), m_c(k)\}} \quad (37)$$

where $Z_{\{m_v(i, j, k), m_a(i), m_b(j), m_c(k)\}}$ is the partition function of the dimer model on a matchgrid with signature $m_v(i, j, k), m_a(i), m_b(j), m_c(k)$. Assume the original matchgates u_a, u_b, u_c have weights

$$m_l = \begin{pmatrix} 0 & a_2^l & b_2^l & 0 & c_2^l & 0 & 0 & d_2^l \end{pmatrix}^t, \quad l = a, b, c$$

see Figure 12. We require

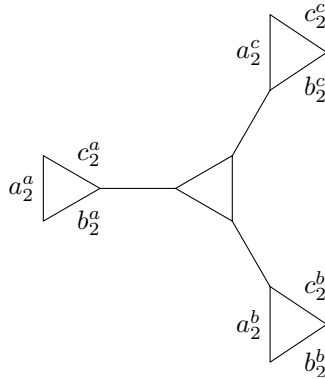


Figure 12: Weighted Matchgate

$$\begin{aligned} m_a(1) &= \begin{pmatrix} 0 & 0 & 0 & 0 & c_2^a & 0 & 0 & d_2^a \end{pmatrix}^t \\ m_a(0) &= \begin{pmatrix} 0 & a_2^a & b_2^a & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t \\ m_b(1) &= \begin{pmatrix} 0 & 0 & b_2^b & 0 & 0 & 0 & 0 & d_2^b \end{pmatrix}^t \\ m_b(0) &= \begin{pmatrix} 0 & a_2^b & 0 & 0 & c_2^b & 0 & 0 & 0 \end{pmatrix}^t \\ m_c(1) &= \begin{pmatrix} 0 & a_2^c & 0 & 0 & 0 & 0 & 0 & d_2^c \end{pmatrix}^t \\ m_c(0) &= \begin{pmatrix} 0 & 0 & b_2^c & 0 & c_2^c & 0 & 0 & 0 \end{pmatrix}^t. \end{aligned}$$

Notice that for any $l \in \{a, b, c\}$, $m_l(0)$ corresponds to configurations with an unoccupied l -edge; and $m_l(1)$ corresponds to configurations with an occupied l -edge. We are trying to determine the new base change matrices on incident edges of v , for each $r_v, r_a(i), r_b(j), r_c(k)$, such that (36), (37), (38) are satisfied simultaneously.

Assume on the adjacent edges of v_a, v_b, v_c we have original base change matrix

$$T_k^j = \begin{pmatrix} n_{0k}^j & n_{1k}^j \\ p_{0k}^j & p_{1k}^j \end{pmatrix} \quad k = 1, 2, 3; \quad j = a, b, c$$

On an a-type edge adjacent to v , we consider 2 possible base change matrices

$$S_1^0 = \begin{pmatrix} m_{01}^0 & 0 \\ q_{01}^0 & q_{11}^0 \end{pmatrix} \quad S_1^1 = \begin{pmatrix} 0 & m_{11}^1 \\ q_{01}^1 & q_{11}^1 \end{pmatrix}$$

On a b-type edge adjacent to v , we consider 2 possible base change matrices

$$S_2^0 = \begin{pmatrix} m_{02}^0 & m_{12}^0 \\ q_{02}^0 & 0 \end{pmatrix} \quad S_2^1 = \begin{pmatrix} m_{02}^1 & m_{12}^1 \\ 0 & q_{12}^1 \end{pmatrix}$$

On a c-type edge adjacent to v , we consider 2 possible base change matrices

$$S_3^0 = \begin{pmatrix} m_{03}^0 & m_{13}^0 \\ q_{03}^0 & 0 \end{pmatrix} \quad S_3^1 = \begin{pmatrix} m_{03}^1 & m_{13}^1 \\ 0 & q_{13}^1 \end{pmatrix}$$

For signatures $r_v, r_a(i), r_b(j), r_c(k)$, we choose basis S_1^i, S_2^j, S_3^k on edges a, b, c . By the realizability condition at vertex v , we have

$$m_v(i, j, k) = (S_1^i \otimes S_2^j \otimes S_3^k)^t \cdot r_v = e_{\{ijk\}} w_{ijk},$$

where

$$w_{ijk} = m_{i1}^i q_{j2}^j q_{k3}^k y_4,$$

and $e_{\{ijk\}}$ is the 8 dimensional vector with entry 1 at the position labeled by the binary sequence $\{ijk\}$, and 0 elsewhere. Obviously the choice of new base change matrices, namely, the position of zeros in the new base change matrices, results in the single configuration $\{ijk\}$ at the matchgate corresponding to v . Now the problem is to determine the entries of the base change matrices satisfying the equations.

According to realizability conditions at vertices v_a, v_b, v_c ,

$$r_a = \sum_{i=1}^2 r_a(i) = \sum_{i=1}^2 S_1^i \otimes T_2^a \otimes T_3^a \cdot m_a(i) \quad (38)$$

$$r_b = \sum_{j=1}^2 r_b(j) = \sum_{j=1}^2 T_1^b \otimes S_2^j \otimes T_3^b \cdot m_b(j) \quad (39)$$

$$r_c = \sum_{k=1}^2 r_c(k) = \sum_{k=1}^2 T_1^c \otimes T_2^c \otimes S_3^k \cdot m_c(k). \quad (40)$$

For the original basis, we have the realizability condition as follows:

$$r_l = T_1^l \otimes T_2^l \otimes T_3^l \cdot m_l. \quad (41)$$

Substituting r_a, r_b, r_c by (41) in (40) we have the following system of linear equations with respect to entries of S

$$\begin{aligned} T_2^a \otimes T_3^a \begin{pmatrix} m_{11}^1 c_2^a \\ m_{01}^2 a_2^a \\ m_{01}^2 b_2^a \\ m_{11}^1 d_2^a \end{pmatrix} &= \begin{pmatrix} x_1^a \\ x_2^a \\ x_3^a \\ x_4^a \end{pmatrix} = T_2^a \otimes T_3^a \begin{pmatrix} n_{11}^a c_2^a \\ n_{01}^a a_2^a \\ n_{01}^a b_2^a \\ n_{11}^a d_2^a \end{pmatrix} \\ T_1^b \otimes T_3^b \begin{pmatrix} q_{12}^1 b_2^b \\ q_{02}^2 a_2^b \\ q_{02}^2 c_2^b \\ q_{12}^1 d_2^b \end{pmatrix} &= \begin{pmatrix} x_3^b \\ x_4^b \\ x_7^b \\ x_8^b \end{pmatrix} = T_1^b \otimes T_3^b \begin{pmatrix} p_{12}^b b_2^b \\ p_{02}^b a_2^b \\ p_{02}^b c_2^b \\ p_{12}^b d_2^b \end{pmatrix} \\ T_1^c \otimes T_2^c \begin{pmatrix} q_{13}^1 a_2^c \\ q_{03}^2 b_2^c \\ q_{03}^2 c_2^b \\ q_{13}^1 d_2^b \end{pmatrix} &= \begin{pmatrix} x_2^c \\ x_4^c \\ x_6^c \\ x_8^c \end{pmatrix} = T_1^c \otimes T_2^c \begin{pmatrix} p_{13}^c a_2^c \\ p_{03}^c b_2^c \\ p_{03}^c c_2^b \\ p_{13}^c d_2^b \end{pmatrix} \end{aligned}$$

Under the assumption that original base change matrices are invertible, we have

$$\begin{aligned} m_{11}^1 &= n_{11}^a, & m_{01}^2 &= n_{01}^a \\ q_{12}^1 &= p_{12}^b, & q_{02}^2 &= p_{02}^b \\ q_{13}^1 &= p_{13}^c, & q_{03}^2 &= p_{03}^c \end{aligned}$$

Therefore, we only need to choose the nonzero entries of the new base change matrices to be the same as the original ones.

For the other seven configurations at vertex v , we can use the same method to achieve a similar result. By splitting each one of r_a, r_b, r_c into 2 parts, we express the partition function of the satisfying assignments as the sum of 8 terms, each one of which is the partition function of the relations with signature $r_v, r_a(i), r_b(j), r_c(k)$, $i, j, k \in \{0, 1\}$. We apply the holographic reduction to each part separately, and we derive that the partition function of the relations with signature $r_v, r_a(i), r_b(j), r_c(k)$, $i, j, k \in \{0, 1\}$ is equal to the partition function of the dimer configurations on the corresponding matchgrids with signature $m_v(i, j, k), m_a(i), m_b(j), m_c(k)$, which corresponds to the local configuration $\{ijk\}$. This may not be a dimer configuration; to make it a dimer configuration, let us divide those 8 terms into two groups: one consists of all $\{i, j, k\}$ satisfying $i + j + k \equiv 0 \pmod{2}$, namely $\{000\}, \{011\}, \{101\}, \{110\}$; the other consists of all $\{i, j, k\}$ satisfying $i + j + k \equiv 1 \pmod{2}$. The sum of partition functions in the first group is equal to the partition function of dimer configurations in a matchgrid with weights illustrated in the left graph of Figure 11, where $t = \frac{w_{000}}{w_{011} + w_{101} + w_{110}}$; the sum of partition functions corresponding to $\{001\}, \{010\}, \{100\}, \{111\}$ is equal to the partition function of dimer configurations in a matchgrid with weights as illustrated in the right graph of Figure 11, up to a multiplication constant w_{111} .

However, in the left graph we change the parity of the total number of vertices, as a result, there is no dimer. Therefore we have the following theorem

Theorem 5.1. *Assume we have a vertex model \mathcal{H}_0 on a finite hexagonal lattice, which is holographic equivalent to the dimer model on a Fisher graph \mathcal{F}_0 . If at a*

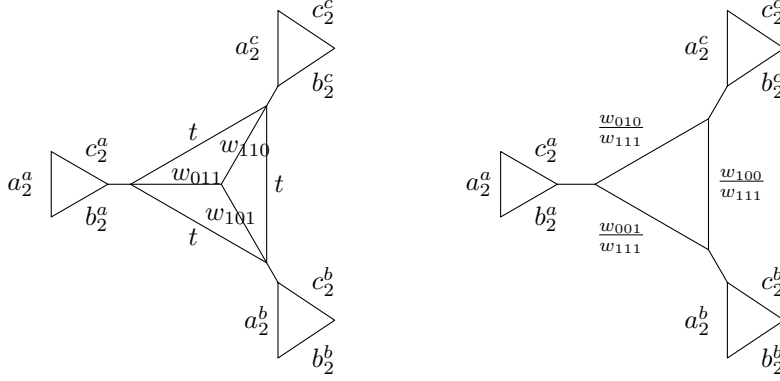


Figure 13: Even and Odd Matchgates

fixed interior vertex v of \mathcal{H}_0 , only one local configuration is allowed, then the partition function of the new vertex model \mathcal{H}_1 can be computed by the partition function of the dimer model on a Fisher graph \mathcal{F}_1 , multiplied by a non-zero constant. \mathcal{F}_0 and \mathcal{F}_1 have the same edge weights except for the matchgate corresponding to v .

5.2 Configuration at Finitely Many Vertices

We are interested in the probability that one specified configuration is allowed at each of the given finitely many vertices for a realizable vertex model on a hexagonal lattice. For simplicity, assume all of them are black. Let V_0^b be the set of all black vertices with specified configuration. To compute the probability, we first give a criterion to construct the new base change matrices on incident edges of V_0^b according to whether the edge is present in the dimer configuration and in the vertex model configuration. Assume $\begin{pmatrix} n_0 & n_1 \\ p_0 & p_1 \end{pmatrix}$ is the original base change matrix on the edge,

Base Change Matrices	Presence in Vertex Configuration	Presence in Dimer
$\begin{pmatrix} 0 & n_1 \\ p_0 & p_1 \end{pmatrix}$	No	Yes
$\begin{pmatrix} n_0 & 0 \\ p_0 & p_1 \end{pmatrix}$	No	No
$\begin{pmatrix} n_0 & n_1 \\ 0 & p_1 \end{pmatrix}$	Yes	Yes
$\begin{pmatrix} n_0 & n_1 \\ p_0 & 0 \end{pmatrix}$	Yes	No

whether the edges incident to vertices in V_0^b are present in the configuration of the vertex model is known, given all the specified configurations at V_0^b . As before, we are going to split the signature of each adjacent white vertex into several parts, and we apply the holographic reduction to each parts separately. Our expectation is that after the reduction process, each part will be equivalent to the partition function of a

single local dimer configuration. The new base change matrix on each edge is chosen according to whether we want the edge to be present in the dimer configuration or not after the reduction. As before, all the non-vanishing entries of the new bases change matrices are equal to the original ones; we will see how it works below.

Consider an arbitrary white vertex $w \in \Gamma(V_0^b)$, the neighbors of V_0^b . Assume r_w is the vertex signature at w , and m_w is the original dimer signature at the corresponding matchgate. Assume

$$m_w = \begin{pmatrix} 0 & a_w & b_w & 0 & c_w & 0 & 0 & d_w \end{pmatrix}^t$$

Let $D(w)$ denote the number of adjacent vertices of w with specified configuration. We classify $\Gamma(V_0^b)$ according to D .

If $D(w) = 1$, we split $r_w = r_w(1) + r_w(2)$. Without loss of generality, assume the left-digit edge(a-type edge, horizontal edge) connects w with $b \in V_0^b$, and the edge wb is present in the configuration of the vertex model. Assume

$$T_1 = \begin{pmatrix} n_{01} & n_{11} \\ p_{01} & p_{11} \end{pmatrix} \quad T_2 = \begin{pmatrix} n_{02} & n_{12} \\ p_{02} & p_{12} \end{pmatrix} \quad T_3 = \begin{pmatrix} n_{03} & n_{13} \\ p_{03} & p_{13} \end{pmatrix}$$

are original base change matrices on the edges adjacent to w . According to the table, the presence of wb in the vertex model configuration implies two possible choices of the new basis on the horizontal edge, namely,

$$S_1^1 = \begin{pmatrix} n_{01} & n_{11} \\ p_{01} & 0 \end{pmatrix} \quad S_1^2 = \begin{pmatrix} n_{01} & n_{11} \\ 0 & p_{11} \end{pmatrix}$$

while on the two other incident edges of w we have the original bases T_2, T_3 . Assume

$$\begin{aligned} m_w(1) &= \begin{pmatrix} 0 & a_w & b_w & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ m_w(2) &= \begin{pmatrix} 0 & 0 & 0 & 0 & c_w & 0 & 0 & d_w \end{pmatrix} \end{aligned}$$

Notice that the nonzero entries $m_w(1)$ have indices $\{001\}$ and $\{010\}$, which corresponds to configurations without the a-edge present; the non-vanishing entries of $m_w(2)$ have indices $\{100\}$ and $\{111\}$, which corresponds to configurations with the a-edge present. Choose

$$\begin{aligned} r_w(1) &= S_1^1 \otimes T_2 \otimes T_3 \cdot m_w(1) \\ r_w(2) &= S_1^2 \otimes T_2 \otimes T_3 \cdot m_w(2) \end{aligned}$$

Then we can check

$$r_w(1) + r_w(2) = \otimes_{i=1}^3 T_i \cdot m_w = r_w$$

If $D(w) = 2$, we split $r_w = \sum_{i=1}^4 r_w(i)$. Without loss of generality, assume the middle-digit(b-type edge) and right-digit(c-type edge) edges connects w to $b_2, b_3 \in V_0^b$, wb_2 is present while wb_3 is not present in the configuration of the vertex model.

According to our criteria listed in the table, on wb_2 we have two different base change matrices

$$S_2^1 = \begin{pmatrix} n_{02} & n_{12} \\ p_{02} & 0 \end{pmatrix} \quad S_2^2 = \begin{pmatrix} n_{02} & n_{12} \\ 0 & p_{12} \end{pmatrix}$$

On wb_3 we have two different base change matrices

$$S_3^1 = \begin{pmatrix} n_{03} & 0 \\ p_{03} & p_{13} \end{pmatrix} \quad S_3^2 = \begin{pmatrix} 0 & n_{13} \\ p_{03} & p_{13} \end{pmatrix}$$

while on the left-digit edge, we keep original basis T_1 , the original basis on the middle and right digit edge are T_2, T_3 , as in the $D(w) = 1$ case. Assume

$$\begin{aligned} m_w(1) &= (0 \ a_w \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \\ m_w(2) &= (0 \ 0 \ b_w \ 0 \ 0 \ 0 \ 0 \ 0) \\ m_w(3) &= (0 \ 0 \ 0 \ 0 \ c_w \ 0 \ 0 \ 0) \\ m_w(4) &= (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ d_w) \end{aligned}$$

Obviously, $m_w(4)$ corresponds to both edge being present, $m_w(3)$ corresponds to neither being present, $m_w(2)$ corresponds to b-edge being present and c-edge not, and $m_w(1)$ corresponds to b-edge not present and c-edge present. Again according to the table, choose

$$\begin{aligned} r_w(1) &= T_1 \otimes S_2^1 \otimes S_3^2 \cdot m_w(1) \\ r_w(2) &= T_1 \otimes S_2^2 \otimes S_3^1 \cdot m_w(2) \\ r_w(3) &= T_1 \otimes S_2^1 \otimes S_3^1 \cdot m_w(3) \\ r_w(4) &= T_1 \otimes S_2^2 \otimes S_3^2 \cdot m_w(4) \end{aligned}$$

we can check

$$\sum_{i=1}^4 r_w(i) = \otimes_1^3 T_i \cdot m_w = r_w$$

If $D(w) = 3$, we split $r_w = \sum_{i=1}^4 r_w(i)$, and assume $m_w(i), i = 1, \dots, 4$ as in $D(w) = 2$. Without loss of generality, assume S_1^1, S_1^2 as in $D(w) = 1$, $S_2^1, S_2^2, S_3^1, S_3^2$ as in $D(w) = 2$; that is, we have vertex-model signature $\{110\}$ at w . Choose

$$\begin{aligned} r_w(1) &= S_1^1 \otimes S_2^1 \otimes S_3^2 \cdot m_w(1) \\ r_w(2) &= S_1^1 \otimes S_2^2 \otimes S_3^1 \cdot m_w(2) \\ r_w(3) &= S_1^2 \otimes S_2^1 \otimes S_3^1 \cdot m_w(3) \\ r_w(4) &= S_1^2 \otimes S_2^2 \otimes S_3^2 \cdot m_w(4) \end{aligned}$$

again we can check $r_w = \sum_{i=1}^4 r_w(i)$ as in $D(w) = 2$.

For all the other local configurations, the same technique works, and we will have a similar result. Assume $V_0^b = \{v_1, \dots, v_p\}$, $\Gamma(V_0^b) = \{w_1, \dots, w_k\}$, then

$$\begin{aligned} S(\tilde{r}_{v_1}, \dots, \tilde{r}_{v_p}) &= [\otimes_{j=1}^k r_{w_j} \otimes \otimes_{w \notin \Gamma(V_0^b)} r_w] \cdot [\otimes_{q=1}^p \tilde{r}_{v_q} \otimes \otimes_{b \notin V_0^b} r_b] \\ &= \sum_{i_1, \dots, i_k} [\otimes_{j=1}^k r_{w_j}(i_j) \otimes \otimes_{w \notin \Gamma(V_0^b)} r_w] \cdot [\otimes_{q=1}^p \tilde{r}_{v_q} \otimes \otimes_{b \notin V_0^b} r_b] \end{aligned}$$

where \tilde{r}_{v_q} is the specified configuration at the vertex v_q . The first equality follows from the definition of the partition function of satisfying assignments. When we compute the tensor product of relation signatures of all black (white) vertices, we get a vector of dimension $2^{|E(G)|}$, where $|E(G)|$ is the number of edges in the planar finite graph G . This vector can be indexed by binary sequences of length $|E(G)|$. Each binary sequence corresponds to a configuration, and the entry there is the product of weights of the local configurations obtained from the configuration restricted to each black (white) vertices. Obviously the inner product of the vector at black vertices and the vector at white vertices are exactly the partition function of the satisfying assignments. The second equality follows from the multi-linearity of the tensor product. For $1 \leq j \leq k$, if $D(w_j) = 1$, $i_j \in \{1, 2\}$; if $D(w_j) = 2, 3$, $i_j \in \{1, 2, 3, 4\}$. For each summand, we choose a basis on incident edges of vertices in V_0^b according to whether the edge is present in the dimer configuration and relation configuration, and keep the original bases on all the other edges. as described above. This is realizable because for each $b \in V_0^b$, whether its incident edges are occupied by dimers are completely determined in each part of the sum. In other words, each term on the right side of the second equality corresponds to an configuration on all the edges incident to vertices in V_0^b , which is a local dimer configuration at each odd matchgate corresponding to white vertices w 's with $D(w) = 3$, To make them be dimer configurations at each black matchgate, we divide those configurations into groups according to the following criterion: two configurations are in the same group if and only if the parity of the number of occupied incident edges at each black vertex in V_0^b is the same. If zero or two incident edges are occupied, we construct an even matchgate with modified weights at the black vertex; if one or three incident edges are occupied, we construct an odd matchgate with modified weights. Notice that the modified weights depend only on the local configuration on the incident edges of the vertex. Since $|V_0^b| = p$, we have 2^p different constructions in total, but only 2^{p-1} of them admit dimer cover, each of which has even number of even matchgates. Therefore we have

Theorem 5.2. *Assume we have a realizable vertex model \mathcal{H}_0 on a finite hexagonal lattice, which is holographic equivalent to the dimer model on a Fisher graph \mathcal{F}_0 . Let V_0^b be a subset of black vertices. If for each $v \in V_0^b$, we only allow one local configuration, this way we obtain a new vertex model \mathcal{H}_1 . The partition function of \mathcal{H}_1 is equal to the sum of partition functions of 2^{p-1} dimer models on matchgrids $\mathcal{F}_1, \dots, \mathcal{F}_{2^{p-1}}$, each of which has an even number of even matchgates. $\mathcal{F}_i (1 \leq i \leq 2^{p-1})$ are the same \mathcal{F}_0 , except for the matchgates corresponding to vertices in V_0^b .*

5.3 Ising Model and Vertex Model

Consider the Ising model on a finite Kagome lattice, embedded into a torus, as illustrated in the following figure.

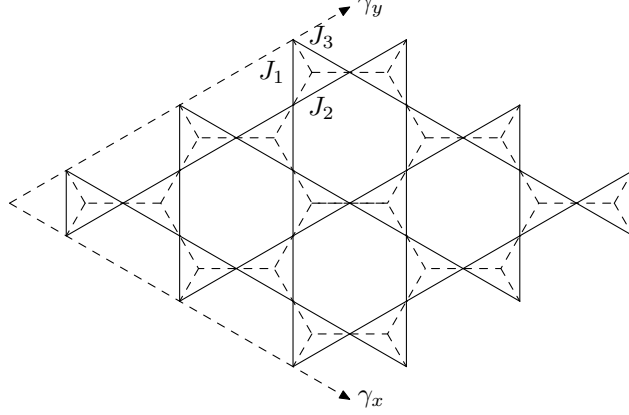


Figure 14: Kagome Lattice and Honeycomb Lattice

The associated honeycomb lattice is illustrated in the Figure by the dashed line. Each vertex of the Kagome lattice corresponds to an edge of the honeycomb lattice; hence each Ising spin configuration on the Kagome lattice corresponds to an edge subset of the honeycomb lattice. If the spin is "+", then the corresponding edge is included in the subset; otherwise the edge is not included. Assume the bonds of the Kagome lattice have interactions J_1, J_2, J_3 , as illustrated in Figure 14. We define a vertex model on the honeycomb lattice with signature at all vertices as follows.

$$\begin{pmatrix} r_{v,000} \\ r_{v,001} \\ r_{v,010} \\ r_{v,011} \\ r_{v,100} \\ r_{v,101} \\ r_{v,110} \\ r_{v,111} \end{pmatrix} = \begin{pmatrix} e^{2(J_1+J_2+J_3)} \\ e^{2J_3} \\ e^{2J_2} \\ e^{2J_1} \\ e^{2J_1} \\ e^{2J_2} \\ e^{2J_3} \\ e^{2(J_1+J_2+J_3)} \end{pmatrix}$$

This way the probability measure of the Ising model on the Kagome lattice is equivalent to the probability measure of the vertex model on the honeycomb lattice. It is trivial to check that this vertex model is orthogonally realizable.

6 Asymptotic Behavior

In this section, we prove the main theorems concerning the asymptotic behavior of realizable vertex models on the periodic hexagonal lattice, as stated in the introduction. Consider an infinite periodic graph G , with period 1×1 , see Figure 9. Our technique to deal with such a graph is to consider a graph G_n with n^2 1×1 fundamental domains, embed G_n into a torus, and consider the limit when $n \rightarrow \infty$, see Page

5. Our first theorem is about the free energy of the infinite periodic hexagonal lattice.

Proof of Theorem 1.1 Assume M_n is the corresponding matchgrid with respect to G_n , and $P_n(z, w)$ is the characteristic polynomial. Obviously M_n is also a quotient graph of a periodic infinite graph modulo a subgraph of \mathbb{Z}^2 generated by $(n, 0)$ and $(0, n)$. The corresponding Kasteleyn matrices here are defined given the orientation of Figure 9. For even n , the crossing orientation can be obtained from the orientation of Figure 9 by reversing all the z -edges and w -edges. By Theorem 3.6, M_n is a Fisher graph, and the partition of the vertex model can be expressed as follows according to the principle of holographic reduction:

$$S(G_n) = Z(M_n) = \frac{1}{2} | -\text{Pf}K_n(1, 1) + \text{Pf}K_n(1, -1) + \text{Pf}K_n(-1, 1) + \text{Pf}K_n(-1, -1) |$$

Thus

$$\max_{u, v \in \{-1, 1\}} |\text{Pf}P_n(u, v)| \leq Z(M_n) \leq 2 \max_{u, v \in \{-1, 1\}} |\text{Pf}P_n(u, v)|$$

On the other hand, according to the formula of enlarging fundamental domains,

$$\frac{1}{n^2} \log \max_{u, v \in \{-1, 1\}} |\text{Pf}K_n(u, v)| = \max_{u, v \in \{-1, 1\}} \frac{1}{2n^2} \sum_{z^n=u} \sum_{w^n=v} \log |P(z, w)|$$

By Theorem 4.4, either $P(z, w)$ has no zeros on \mathbb{T}^2 , or it has a single real node, in which case any sample point in $\max_{u, v \in \{-1, 1\}} \frac{1}{2n^2} \sum_{z^n=u} \sum_{w^n=v} \log |P(z, w)|$ is at least $\frac{C}{n}$ from the real node, for some constant $C > 0$. Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^2} S(G_n) &= \lim_{n \rightarrow \infty} \max_{u, v \in \{-1, 1\}} \frac{1}{2n^2} \sum_{z^n=u} \sum_{w^n=v} \log |P(z, w)| \\ &= \frac{1}{8\pi^2} \iint_{|z|=1, |w|=1} \log P(z, w) \frac{dz}{iz} \frac{dw}{iw} \end{aligned}$$

□

For each G_n , a measure λ_n is defined as in (1). Assume at a fixed vertices v , we only allow configuration c_v . Let μ_n be the Boltzmann measure of dimer configurations on M_n . Let \tilde{M}_n be the matchgrid corresponding to the vertex model which only allows c_v at v , as described in Theorem 5.1. Let m_v be the matchgate of \tilde{M}_n , corresponding to v , and d_j be a local dimer configuration at m_v , w_{d_j} be product of weights of matchgate edges included in d_j , and V_{d_j} be the set of external vertices of matchgates m_v occupied by dimer configuration d_j . In our graph, every matchgate has 3 external vertices. Our second theorem is about the asymptotic behavior of the measure λ_n .

Proof of Theorem 1.2 According to Theorem 5.1

$$\begin{aligned}\lambda_n(c_1, \dots, c_p) &= \frac{Z(\tilde{M}_n)}{Z(M_n)} = \sum_{d_j} \frac{Z(\tilde{M}_n(d_j))}{Z(M_n)} \\ &= \sum_{d_j} w_{d_j} \frac{Z(M_n \setminus V(d_j))}{Z(M_n)}\end{aligned}$$

where the sum is over all local dimer configurations d_j . Since the number of local configurations is finite, it suffices to consider $\lim_{n \rightarrow \infty} \frac{Z(M_n \setminus V(d_j))}{Z(M_n)}$. Note that \tilde{M}_n differs from M_n only on edge weights of m_v , hence $M_n \setminus V(d_j)$ and $\tilde{M}_n \setminus V(d_j)$ are the same.

Given d_j , let $W_{n,d_j} = M_n \setminus V(d_j)$ be the subgraph of M_n by removing all vertices occupied by the configuration (d_j) , as well as their incident edges. Then

$$\begin{aligned}\frac{Z(W_{n,d_j})}{Z(M_n)} &= \frac{1}{2Z_n} | -\text{Pf}(K_n^{1,1}(W_{n,d_j})) + \text{Pf}(K_n^{-1,1}(W_{n,d_j})) \\ &\quad + \text{Pf}(K_n^{1,-1}(W_{n,d_j})) + \text{Pf}(K_n^{-1,-1}(W_{n,d_j})) | \end{aligned}$$

First of all, let us assume $P(z, w)$ has no zeros on \mathbb{T}^2 . According to the formula of enlarging fundamental domains, for any $(\theta, \tau) \in \{-1, 1\}$, $\text{Pf}(K_n^{\theta, \tau}) \neq 0$. Then

$$|\text{Pf}(K_n^{\theta, \tau})_{E^c}| = |\text{Pf}(K_n^{\theta, \tau})_E^{-1}| |\text{Pf}(K_n^{\theta, \tau})|$$

In [3, 8], it was proved that given two vertices (u, x_1, y_1) and (v, x_2, y_2)

$$(K_n^{\theta, \tau})^{-1}((u, x_1, y_1), (v, x_2, y_2)) = \frac{1}{n^2} \sum_{z^n = \theta} \sum_{w^n = \tau} z^{x_1 - x_2} w^{y_1 - y_2} \frac{\text{Cof}(K(z, w))_{u,v}}{P(z, w)}$$

Since $P(z, w)$ has no zeros on \mathbb{T}^2 , we have

$$\begin{aligned}\lim_{n \rightarrow \infty} (K_n^{\theta, \tau})^{-1}((u, x_1, y_1), (v, x_2, y_2)) &= \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} z^{x_1 - x_2} w^{y_1 - y_2} \frac{\text{Cof}(K(z, w))_{u,v}}{P(z, w)} \frac{dz}{iz} \frac{dw}{iw} \\ &= K_\infty^{-1}((u, x_1, y_1), (v, x_2, y_2))\end{aligned}$$

As $n \rightarrow \infty$, each entry of $(K_n^{\theta, \tau})^{-1}$ is convergent, so is the Pfaffian of a finite order sub-matrix $(K_n^{\theta, \tau})_{V(d_j)}^{-1}$, and we have

$$\lim_{n \rightarrow \infty} \frac{Z(M_n \setminus V(d_j))}{Z(M_n)} = |\text{Pf}(K_\infty^{-1})_{V(d_j)}|$$

If $P(z, w)$ has a real node on \mathbb{T}^2 , without loss of generality, we can assume the real node is $(1, 1)$. It was proved in [1] that if $P(z, w) = 0$ has a node at $(1, 1)$, then for any fixed finite subset E

$$\lim_{n \rightarrow \infty} \frac{1}{2Z_n} [\text{Pf}(K_n^{-1,1})_{E^c} + \text{Pf}(K_n^{1,-1})_{E^c} + \text{Pf}(K_n^{-1,-1})_{E^c}] = \text{Pf}(K_\infty^{-1})_E$$

and

$$\lim_{n \rightarrow \infty} \frac{\text{Pf}(K_n^{1,1})_{E^c}}{2Z_n} = 0$$

If we take $E = V(d_{i,j}, \dots, d_{p,j})$, our theorem follows. \square

7 Simulation of 1-2 Model

Assume we have 1-2 model with signature $r_s = (0 \ c \ b \ a \ a \ b \ c \ 0)^t$ at all vertices. In other words, at each vertex, either one or two edges are allowed to be present in a local configuration. By Section 3.2, the signature is orthogonally realizable with base change matrix $T = \begin{pmatrix} \cos \frac{3\pi}{4} & \sin \frac{3\pi}{4} \\ -\sin \frac{3\pi}{4} & \cos \frac{3\pi}{4} \end{pmatrix}$ on all edges.

We define a discrete-time, time-homogeneous Markov chain \mathfrak{M}_t with state space the set of all configurations of the 1-2 model. For an $n \times n$ honeycomb lattice embedded into a torus, the state space is finite, and let us denote it by $\{i_k\}_{k=1}^K$. Let $\Gamma(i_k)$ be the set of configurations that can be obtained from configuration i_k by adding or deleting a single edge. Assume p, q, r are edge variables, namely $p, q, r \in \{a, b, c\}$, and $\{p, q, r\} = \{a, b, c\}$. Define $\Gamma_{p,+q}(i_k)$ be the set of configurations of 1-2 model which can be obtained from i_k by adding a single q -edge uv ; before adding uv , only a p -edge is present at both u and v . Define $\Gamma_{p,-q}(i_k)$ be the set of configurations of 1-2 model which can be obtained from i_k by deleting a single q -edge uv ; after deleting uv , only a p -edge is present at both u and v . Define $\Gamma_0(i_k) = \Gamma(i_k) \setminus \{\cup_{p,q \in a,b,c, p \neq q} \Gamma_{p,+q}(i_k) \cup_{p,q \in a,b,c, p \neq q} \Gamma_{p,-q}(i_k)\}$. Define entries of transition matrix for \mathfrak{M}_t as follows:

$$P(i_l|i_k) = \begin{cases} \frac{1}{3n^2} & \text{if } i_l \in \Gamma_0(i_k) \\ \frac{1}{3n^2} & \text{if } i_l \in \Gamma_{p,+q}(i_k) \text{ and } r \geq p \\ \frac{1}{3n^2} \frac{r^2}{p^2} & \text{if } i_l \in \Gamma_{p,+q}(i_k) \text{ and } r < p \\ \frac{1}{3n^2} & \text{if } i_l \in \Gamma_{p,-q}(i_k) \text{ and } r \leq p \\ \frac{1}{3n^2} \frac{p^2}{r^2} & \text{if } i_l \in \Gamma_{p,-q}(i_k) \text{ and } r > p \\ 1 - \sum_{i_j \in \Gamma(i_k)} P(i_j|i_k) & \text{if } i_l = i_k \\ 0 & \text{else} \end{cases}$$

Obviously, \mathfrak{M}_t is aperiodic. For more information on Markov chain, see [11, 12]. Moreover, we have

Proposition 7.1. \mathfrak{M}_t is irreducible.

Proof. By definition, we only need to prove that any two configurations communicate to each other. We claim that any two dimer configuration can be obtained from each other by finite steps. In fact, the symmetric difference of any two dimer configurations is a union of finitely many loops. Obviously one dimer configuration can be obtained from any other dimer configuration by first adding finitely many edges to achieve their union, then deleting alternating edges of loops. Notice that we have a 1-2 configuration at each step. Hence we only need to prove that any configuration of 1-2 model can reach a dimer by finite steps, each of which is adding or deleting one single edge.

Let us start with an arbitrary configuration of 1-2 model. There are 3 types of connected local configurations: loops with even number of edges; zigzag paths with odd number of edges; zigzag paths with even number of edges. For the first and second types, we can always achieve dimers by deleting alternating edges. Hence we can assume that all the zigzag paths with even number of edges are of length 2,

and all the other connected local configurations are dimers. There are two types of length-2 paths. One has vertices black-white-black (BWB), and the other has vertices white-black-white (WBW). For each fixed configuration, the number of BWB paths is the same as the number of WBW paths; otherwise the complement graph cannot be covered by dimers. Consider an arbitrary WBW path in a fixed configuration, as illustrated in Figure 12.

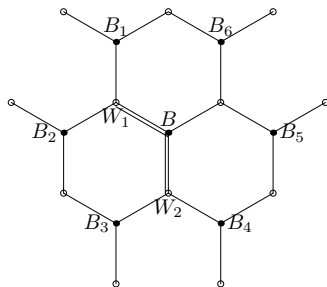


Figure 15: WBW Path

One can easily check that no matter what the configuration is, it communicates with a configuration satisfying one of the following two conditions: 1. the number of length-2 edges is decreased by 2; 2. it can be moved to a WBW configurations containing B_i , for all $1 \leq i \leq 6$. Hence if the number of length-2 paths does not decrease, one can transverse all black vertices without meeting a BWB path, because our graph is finite. However, this is impossible because a BWB path always exists as long as a WBW path exists. \square

For an irreducible, aperiodic Markov chain \mathfrak{M}_t with transition matrix P and stationary distribution π , let x_0 be an arbitrary initial distribution. Then

$$\lim_{n \rightarrow \infty} x_0 P^n = \pi.$$

Therefore, in order to sample a configuration, we can approximately sample according to the distribution $x_0 P^N$, with large N . To that end, first we choose a fixed dimer configuration with probability 1 as the initial distribution x_0 , then we randomly change the configuration by adding or deleting a single edge according to the conditional probability specified by the transition matrix P . If neither adding nor deleting $u_1 u_2$ ends up with a satisfying configuration, we just keep the previous configuration; else we get a new configuration by adding or deleting $u_1 u_2$. Then we repeat the process for N steps. This way we get a sample for distribution $x_0 P^N$, if N is sufficiently large, this is approximately a sample for distribution π , which is exactly the distribution given by 1-2 model.

Example 7.2. (*Uniform 1-2 Model*) Consider the 1-2 model with $a = b = c = 1$. After the holographic reduction, the signature of each matchgate is

$$\left(0 \quad \frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad 0 \quad \frac{\sqrt{2}}{2} \quad 0 \quad 0 \quad -\frac{3\sqrt{2}}{2} \right)'$$

It is gauge equivalent to a positive-weight dimer model on Fisher graph, whose spectral curve does not intersect \mathbb{T}^2 . We are interested in the probability that a $\{001\}$ dimer occurs, that is, at a pair of adjacent vertices v_1, v_2 connected by an a -edge, only the configuration $\{001\}$ is allowed. By the technique described in section 5 and section 6, the partition function of the configurations in which a dimer is present at $v_1 v_2$ is equal to the sum of two partition functions, one corresponds to both v_1 and v_2 are replaced by an odd matchgate with signatures

$$\begin{pmatrix} 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} \end{pmatrix};$$

the other corresponds to both v_1 and v_2 are replaced by an even matchgate with signatures

$$\begin{pmatrix} \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 \end{pmatrix}$$

Both of them have the same partition function as a graph with positive weights. Moreover, if we give a base change matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $v_1 v_2$ edge, and apply holographic reduction, we see that the two graphs with positive weights are holographic equivalent. Hence it suffices to consider the one with a pair of odd matchgates with modified weights. Then

$$Pr(\text{a dimer is present at } v_1 v_2) = \frac{Z_{001}}{18Z}$$

where Z_{001} is the partition function of dimer configurations with weight 1 on triangles corresponding to v_1 and v_2 and weight $\frac{1}{3}$ on all the other triangles, and Z is the partition function with weight $\frac{1}{3}$ on all the triangles. Moreover

$$\begin{aligned} Z_{001} &= Z^{001001} + Z^{001111} + Z^{010010} + Z^{010100} + Z^{100100} + Z^{100010} + Z^{111001} + Z^{111111} \\ &= Z^{001001} + 2Z^{001111} + 2Z^{010010} + 2Z^{010100} + Z^{111111} \end{aligned}$$

where $Z^{ijk, \tilde{ijk}}$ is the partition function of dimer configurations with fixed configuration ijk, \tilde{ijk} on triangles corresponding to v_1 and v_2 . The second equality follows from symmetry. Meanwhile

$$\frac{1}{9}Z^{001001} + \frac{2}{3}Z^{001111} + \frac{2}{9}Z^{010010} + \frac{2}{9}Z^{010100} + Z^{111111} = Z,$$

then

$$\begin{aligned} Pr(\text{a dimer is present at } v_1 v_2) &= \frac{1}{18} \left(1 + \frac{8Z^{001001}}{9Z} + \frac{4Z^{001111}}{3Z} + \frac{16Z^{010010}}{9Z} + \frac{16Z^{010100}}{9Z} \right) \\ &= \frac{1}{18} \left(1 + \frac{4}{3}|(K_\infty^{-1})_{23}| + \frac{4}{9}|\text{Pf}(K_\infty^{-1})_{2356}| + \frac{16}{3}|(K_\infty^{-1})_{13}| \right) \end{aligned}$$

where

$$\begin{aligned}
(K_\infty^{-1})_{23} &= \frac{3}{16\pi^2} \iint_{\mathbb{T}^2} \frac{w(zw + 1 + z - 9w)}{2(z^2 + w^2) + 2(z + w) + 2zw(z + w) - 21zw} \frac{dz}{iz} \frac{dw}{iw} = (K_\infty^{-1})_{56} \\
(K_\infty^{-1})_{13} &= -\frac{3}{16\pi^2} \iint_{\mathbb{T}^2} \frac{w(w + z + z^2 - 9zw)}{2(z^2 + w^2) + 2(z + w) + 2zw(z + w) - 21zw} \frac{dz}{iz} \frac{dw}{iw} \\
(K_\infty^{-1})_{25} &= \frac{1}{16\pi^2} \iint_{\mathbb{T}^2} \frac{8z + 8zw - 82w + 9w^2 + 9}{2(z^2 + w^2) + 2(z + w) + 2zw(z + w) - 21zw} \frac{dz}{iz} \frac{dw}{iw} = (K_\infty^{-1})_{36} \\
(K_\infty^{-1})_{26} &= \frac{1}{16\pi^2} \iint_{\mathbb{T}^2} \frac{-z - zw - w + 9}{2(z^2 + w^2) + 2(z + w) + 2zw(z + w) - 21zw} \frac{dz}{iz} \frac{dw}{iw} = (K_\infty^{-1})_{35}
\end{aligned}$$

The entries of the inverse matrix can be expressed as elliptic functions. The probability that a $\{001\}$ dimer occurs is approximately 6%. A sample of the uniform 1-2 model is illustrated in Figure 16.

Example 7.3. (Critical 1-2 Model) Consider the 1-2 model with $a = 4, b = c = 1$. After the holographic reduction, it has the same partition function as a positive-weight dimer model on Fisher graph, whose spectral curve has a single real node on \mathbb{T}^2 . The probability of the configuration $\{011\}$

$$Pr(\{011\}) = \frac{Z_{011}}{3Z} = \frac{1}{3}(|(K_\infty^{-1})_{12}| + |(K_\infty^{-1})_{23}| + |(K_\infty^{-1})_{13}|) + |\text{Pf}(K_\infty^{-1})_{123456}|$$

Z_{011} is the partition function on a Fisher graph with weights 1, 1, 1 on one triangle, and weights $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$ on all the other triangles. Z is the partition function with weights $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$ on all the triangles. By symmetry $|(K_\infty^{-1})_{12}| = |(K_\infty^{-1})_{23}|$. Then

$$\begin{aligned}
Pr(\{011\}) &= \frac{1}{3} + \frac{2}{9}|(K_\infty^{-1})_{12}| + \frac{2}{9}|(K_\infty^{-1})_{23}| \\
&= \frac{1}{3} + \frac{1}{6\pi^2} \left| \iint_{\mathbb{T}^2} \frac{w(4wz + 4 + z - 9w)}{-7(w^2 + z^2) + 32wz(w + z) + 32(w + z) - 114wz} \frac{dz}{iz} \frac{dw}{iw} \right| \\
&\quad + \frac{1}{3\pi^2} \left| \iint_{\mathbb{T}^2} \frac{z(9wz - 4z - w^2 - 4w)}{-7(w^2 + z^2) + 32wz(w + z) + 32(w + z) - 114wz} \frac{dz}{iz} \frac{dw}{iw} \right| \\
&= \frac{1}{3} + \frac{1}{12\pi} \left| \int_{|z|=1} \frac{8}{32z - 7} - \frac{18}{z(32z - 7)} + \frac{16z^2 - 85z + 29}{z(32z - 7)\sqrt{4z^2 - 17z + 4}} dz \right| \\
&\quad + \frac{1}{3\pi} \left| \int_{|w|=1} \frac{9}{(32w - 7)} - \frac{4}{w(32w - 7)} + \frac{22w^2 - 35w + 8}{w(32w - 7)\sqrt{4w^2 - 17w + 4}} dw \right| \\
&= \frac{23}{48} - \frac{25}{112\pi} \arctan \frac{4}{3} - \frac{65}{336\pi} \arctan \frac{44}{117} \approx 39\%.
\end{aligned}$$

For $\sqrt{\cdot}$ we choose a branch with positive real part. This integral can be evaluated explicitly because the graph has critical edge weights. A sample of the critical 1-2 model is illustrated in Figure 17.

Question: How large should N be as a function of the size of the graph?

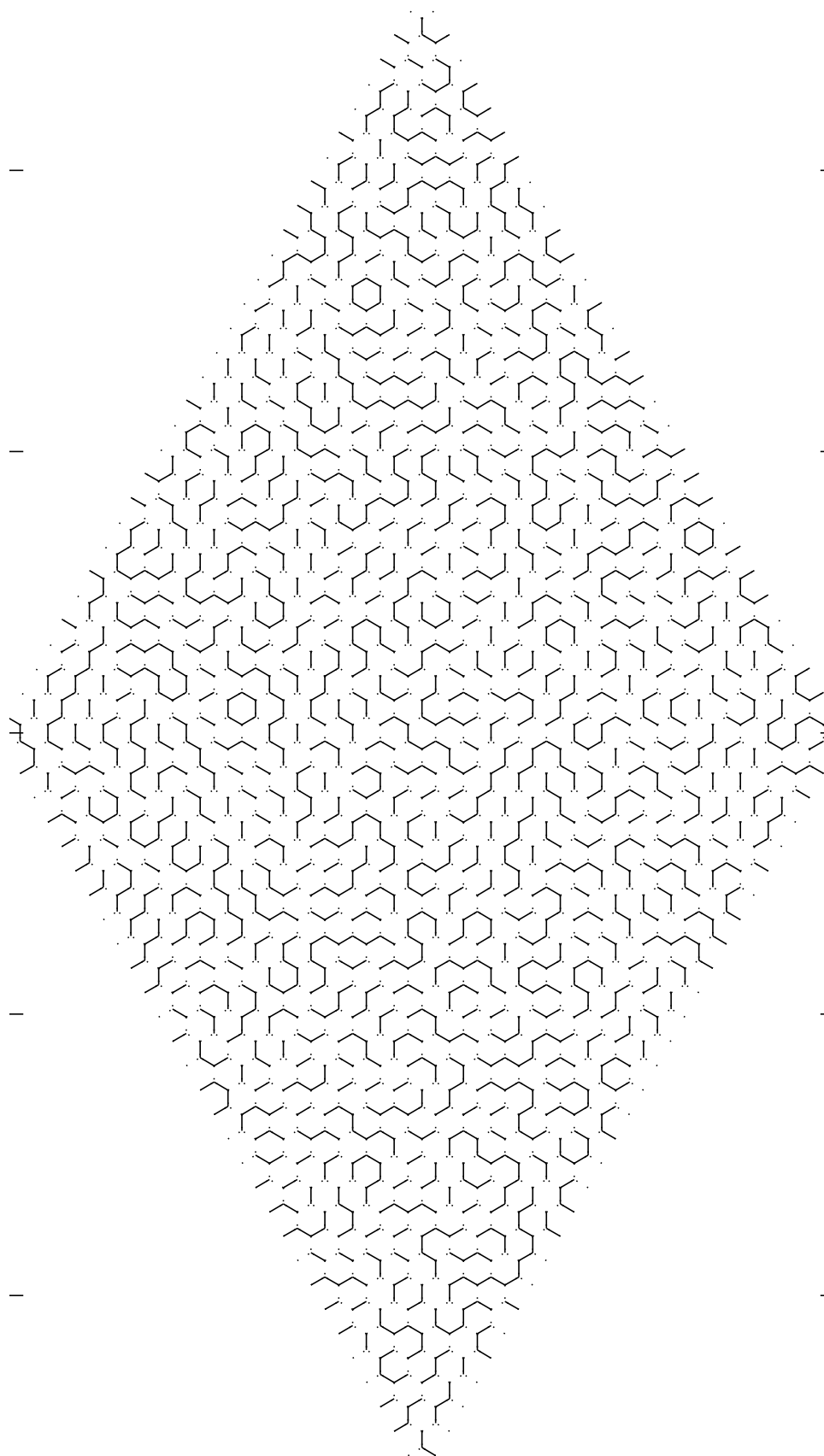


Figure 16: Sample 37 of Uniform 1-2 Model

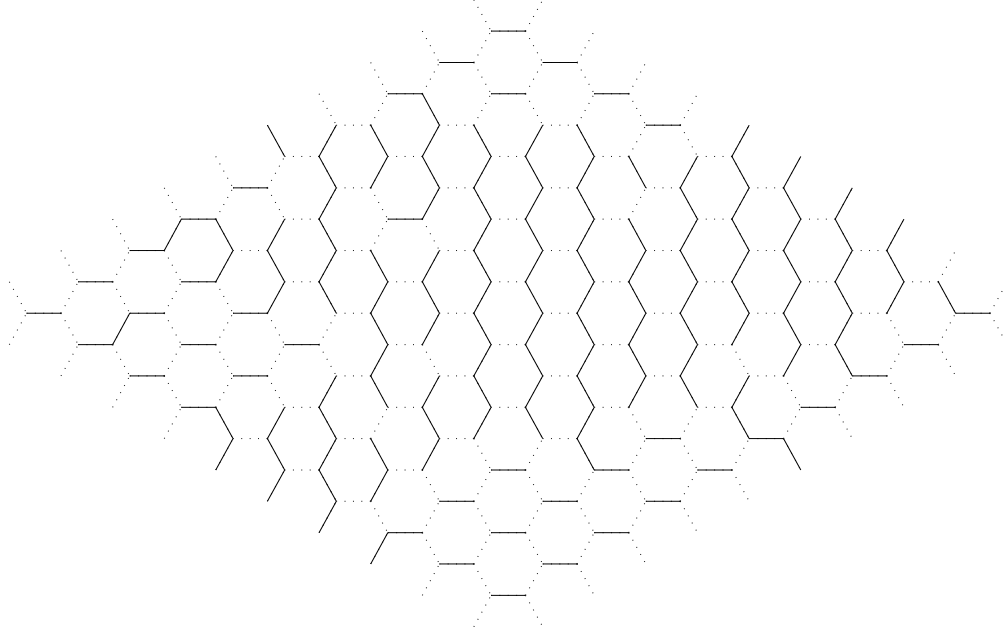


Figure 17: Sample of Critical 1-2 Model

A Realizability Conditions

In this section we give explicit realizability conditions for periodic honeycomb lattice embedded on a $n \times n$ domain. Let us consider an arbitrary vertex x with adjacent vertices y, z, w . Assume relations have following signatures:

$$\begin{aligned} r_x &= (x_1 \quad \dots \quad x_8)^t & r_y &= (y_1 \quad \dots \quad y_8)^t \\ r_z &= (z_1 \quad \dots \quad z_8)^t & r_w &= (w_1 \quad \dots \quad w_8)^t \end{aligned}$$

According to the realizability equation (1), after a lengthy computation, we have

Theorem A.1. *A periodic network of relation on $n \times n$ honeycomb lattice is relizable if and only if at each vertex x with adjacent vertices y, z, w connected by a, b, c edge, respectively, the signatures satisfy the following equation:*

$$\sum_{i,j,k \in \{0,1,2\}} X_{ijk} Y_i Z_j W_k = 0$$

where

$$\begin{aligned} Y_0 &= y_1 y_4 - y_2 y_3 & Y_1 &= y_1 y_8 + y_4 y_5 - y_2 y_7 - y_3 y_6 & Y_2 &= y_5 y_8 - y_6 y_7 \\ Z_0 &= z_1 z_6 - z_2 z_5 & Z_1 &= z_1 z_8 + z_3 z_6 - z_4 z_5 - z_2 z_7 & Z_2 &= z_3 z_8 - z_4 z_7 \\ W_0 &= w_1 w_7 - w_3 w_5 & W_1 &= w_1 w_8 + w_2 w_7 - w_5 w_4 - w_3 w_6 & W_2 &= w_2 w_8 - w_6 w_4 \end{aligned}$$

and

$$\begin{aligned}
X_{000} &= x_1^2(x_3x_6 + x_2x_7 + x_4x_5 - x_1x_8) - 2x_1x_2x_3x_5 \\
X_{001} &= x_1^2(x_6x_4 - x_2x_8) + x_2^2(x_1x_7 - x_3x_5) \\
X_{002} &= x_2^2(x_2x_7 - x_1x_8 - x_3x_6 - x_4x_5) + 2x_1x_2x_4x_6 \\
X_{010} &= x_3^2(x_1x_6 - x_2x_5) + x_1^2(x_7x_4 - x_3x_8) \\
X_{011} &= x_1x_2(x_4x_7 - x_3x_8) + x_3x_4(x_1x_6 - x_2x_5) \\
X_{012} &= x_2^2(x_4x_7 - x_3x_8) + x_4^2(x_1x_6 - x_2x_5) \\
X_{020} &= x_3^2(x_3x_6 - x_2x_7 - x_4x_5 - x_1x_8) + 2x_1x_4x_3x_7 \\
X_{021} &= x_4^2(x_1x_7 - x_3x_5) + x_3^2(x_6x_4 - x_2x_8) \\
X_{022} &= x_4^2(x_2x_7 + x_1x_8 + x_3x_6 - x_4x_5) - 2x_2x_3x_4x_8
\end{aligned}$$

$$\begin{aligned}
X_{100} &= x_5^2(x_1x_4 - x_2x_3) + x_1^2(x_6x_7 - x_5x_8) \\
X_{101} &= x_1x_2(x_6x_7 - x_5x_8) + x_5x_6(x_1x_4 - x_2x_3) \\
X_{102} &= x_6^2(x_1x_4 - x_2x_3) + x_2^2(x_6x_7 - x_5x_8) \\
X_{110} &= x_1x_3(x_7x_6 - x_5x_8) + x_5x_7(x_1x_4 - x_2x_3) \\
X_{111} &= x_1x_4x_6x_7 - x_2x_3x_5x_8 \\
X_{112} &= x_4x_8(x_1x_6 - x_2x_5) + x_2x_6(x_4x_7 - x_3x_8) \\
X_{120} &= x_7^2(x_1x_4 - x_2x_3) + x_3^2(x_6x_7 - x_5x_8) \\
X_{121} &= x_3x_7(x_4x_6 - x_2x_8) + x_4x_8(x_1x_7 - x_3x_5) \\
X_{122} &= x_8^2(x_1x_4 - x_2x_3) + x_4^2(x_6x_7 - x_5x_8)
\end{aligned}$$

$$\begin{aligned}
X_{200} &= x_5^2(x_4x_5 - x_1x_8 - x_3x_6 - x_2x_7) + 2x_1x_5x_6x_7 \\
X_{201} &= x_5^2(x_6x_4 - x_2x_8) + x_6^2(x_1x_7 - x_3x_5) \\
X_{202} &= x_6^2(x_1x_8 + x_2x_7 + x_4x_5 - x_3x_6) - 2x_5x_6x_2x_8 \\
X_{210} &= x_5^2(x_4x_7 - x_3x_8) + x_7^2(x_1x_6 - x_2x_5) \\
X_{211} &= x_6x_8(x_1x_7 - x_3x_5) + x_5x_7(x_4x_6 - x_2x_8) \\
X_{212} &= x_8^2(x_1x_6 - x_2x_5) + x_6^2(x_4x_7 - x_3x_8) \\
X_{220} &= x_7^2(x_1x_8 + x_3x_6 + x_4x_5 - x_2x_7) - 2x_3x_8x_5x_7 \\
X_{221} &= x_8^2(x_1x_7 - x_3x_5) + x_7^2(x_6x_4 - x_2x_8) \\
X_{222} &= x_8^2(x_1x_8 - x_3x_6 - x_2x_7 - x_4x_5) + 2x_7x_8x_6x_4
\end{aligned}$$

Theorem A.2. *A periodic network of relation with $n \times n$ fundamental domain is bipartite realizable if and only if it is realizable and at any vertex v , the signature satisfy*

$$\begin{aligned}
&v_1^2v_8^2 + v_2^2v_7^2 + v_3^2v_6^2 + v_4^2v_5^2 - 2v_1v_8v_2v_7 - 2v_1v_8v_3v_6 - 2v_1v_8v_4v_5 \\
&- 2v_2v_7v_3v_6 - 2v_2v_7v_4v_5 - 2v_3v_6v_4v_5 + 4v_1v_4v_6v_7 + 4v_2v_3v_5v_8 = 0
\end{aligned}$$

Proof. Without loss of generality, we assume at each matchgate, the signature $\{111\}$ is 0, and assume v is a black vertex. The $\{111\}$ entry of the signature of a black vertex is 0 gives us

$$a_{11}a_{12}a_{13}v_1 + a_{11}a_{12}v_2 + a_{11}a_{13}v_3 + a_{11}v_4 + a_{12}a_{13}v_5 + a_{12}v_6 + a_{13}v_7 + v_8 = 0 \quad (42)$$

The parity constraint implies that the $\{110\}$ entry is also 0, namely

$$a_{11}a_{12}a_{03}v_1 + a_{11}a_{12}v_2 + a_{11}a_{03}v_3 + a_{11}v_4 + a_{12}a_{03}v_5 + a_{12}v_6 + a_{03}v_7 + v_8 = 0 \quad (43)$$

From (42)(43), we can solve a_1 , and solution has following form

$$a_{11} = \frac{N_1}{D_1} = \frac{N_2}{D_2}$$

Then $N_1D_2 - N_2D_1 = 0$. Under the assumption that $a_{03} \neq a_{13}$, we have

$$(-v_2v_5 + v_6v_1)a_{12}^2 + (v_8v_1 + v_6v_3 - v_4v_5 - v_2v_7)a_{12} + v_8v_3 - v_4v_7 = 0 \quad (44)$$

From (12), we have

$$2(v_1v_6 - v_2v_5)a_{02}a_{12} + (v_1v_8 + v_3v_6 - v_4v_5 - v_2v_7)(a_{02} + a_{12}) + 2(v_3v_8 - v_4v_7) = 0 \quad (45)$$

$2 \times (44) - (45)$, under the assumption that $a_{02} \neq a_{12}$, we have

$$a_{12} = -\frac{v_1v_8 + v_3v_6 - v_4v_5 - v_2v_7}{2(v_1v_6 - v_2v_5)}$$

which implies that equation (44) has double real roots, and its discriminant is 0. That is exactly the statement of the theorem. \square

Proof of Proposition 3.9 Since holographic reduction is an invertible process, two matchgrids M and \hat{M} are holographically equivalent if and only if there exists a basis for each edge, such that one can be transformed to the other using holographic reduction. Obviously holographic equivalent matchgrids have the same dimer partition function. Assume weights m_b, m_w of M and \hat{m}_b, \hat{m}_w of \hat{M} are as follows:

$$\begin{aligned} m_w^{ij} &= \begin{pmatrix} 0 & c_1^{ij} & b_1^{ij} & 0 & a_1^{ij} & 0 & 0 & d_1^{ij} \end{pmatrix}^t \\ m_b^{ij} &= \begin{pmatrix} 0 & c_2^{ij} & b_2^{ij} & 0 & a_2^{ij} & 0 & 0 & d_2^{ij} \end{pmatrix}^t \\ \hat{m}_w^{ij} &= \begin{pmatrix} 0 & \hat{c}_1^{ij} & \hat{b}_1^{ij} & 0 & \hat{a}_1^{ij} & 0 & 0 & \hat{d}_1^{ij} \end{pmatrix}^t \\ \hat{m}_b^{ij} &= \begin{pmatrix} 0 & \hat{c}_2^{ij} & \hat{b}_2^{ij} & 0 & \hat{a}_2^{ij} & 0 & 0 & \hat{d}_2^{ij} \end{pmatrix}^t \end{aligned}$$

Then by equations (16)-(21), and the uniqueness of the basis on each edge, we have

$$\frac{a_1^{ij} b_1^{ij}}{d_1^{ij} c_1^{ij}} = \frac{d_2^{i,j-1} c_2^{i,j-1}}{b_2^{i,j-1} a_2^{i,j-1}} = a_{03}^{ij} a_{13}^{ij} = b_{03}^{i,j-1} b_{13}^{i,j-1} \quad (46)$$

$$\frac{a_1^{ij} c_1^{ij}}{b_1^{ij} d_1^{ij}} = \frac{d_2^{i-1,j} b_2^{i-1,j}}{a_2^{i-1,j} c_2^{i-1,j}} = a_{02}^{ij} a_{12}^{ij} = b_{02}^{i-1,j} b_{12}^{i-1,j} \quad (47)$$

$$\frac{b_1^{ij} c_1^{ij}}{a_1^{ij} d_1^{ij}} = \frac{d_2^{ij} a_2^{ij}}{b_2^{ij} c_2^{ij}} = a_{01}^{ij} a_{11}^{ij} = b_{01}^{ij} b_{11}^{ij} \quad (48)$$

Plugging in (46)-(48) to (9)-(10), we have

$$\begin{cases} \hat{c}_2^{ij} = c_2^{ij} n_{01}^{i,j,0} n_{02}^{i,j,0} p_{13}^{i,j,0} \cdot C_1^{ij} \\ \hat{b}_2^{ij} = b_2^{ij} n_{01}^{i,j,0} p_{12}^{i,j,0} n_{03}^{i,j,0} \cdot C_1^{ij} \\ \hat{a}_2^{ij} = a_2^{ij} p_{11}^{i,j,0} n_{02}^{i,j,0} n_{03}^{i,j,0} \cdot C_1^{ij} \\ \hat{d}_2^{ij} = d_2^{ij} p_{11}^{i,j,0} p_{12}^{i,j,0} p_{13}^{i,j,0} \cdot C_1^{ij} \end{cases}$$

$$\begin{cases} \hat{c}_1^{ij} = \frac{c_1^{ij}}{n_{01}^{i,j,1} n_{02}^{i,j,1} p_{13}^{i,j,1}} \cdot C_2^{ij} \\ \hat{b}_1^{ij} = \frac{b_1^{ij}}{n_{01}^{i,j,1} p_{12}^{i,j,1} n_{03}^{i,j,1}} \cdot C_2^{ij} \\ \hat{a}_1^{ij} = \frac{a_1^{ij}}{p_{11}^{i,j,1} n_{02}^{i,j,1} n_{03}^{i,j,1}} \cdot C_2^{ij} \\ \hat{d}_1^{ij} = \frac{d_1^{ij}}{p_{11}^{i,j,1} p_{12}^{i,j,1} p_{13}^{i,j,1}} \cdot C_2^{ij} \end{cases}$$

To prove that probability measures of M and \hat{M} are identical, we only need to prove that for any dimer configuration, the products of weights differ by the same constant factor. Each dimer configuration corresponds to a binary sequence of length N , where $N = 3n^2$ is the number of connecting edges. Choose an arbitrary edge with basis $\begin{pmatrix} n_0 & n_1 \\ p_0 & p_1 \end{pmatrix}$. If the edge is occupied, then from M to \hat{M} , adjacent generator weight is divided by p_1 , while adjacent recognizer weight is multiplied by p_1 . If the edge is unoccupied, then from M to \hat{M} , adjacent generator weight is divided by n_0 , while the adjacent recognizer weight is multiplied by n_0 . Therefore, the total effect is for any particular dimer configuration ϖ

$$\frac{\text{Partition}(\varpi \text{ on } \hat{M})}{\text{Partition}(\varpi \text{ on } M)} = \prod_{i,j} C_1^{ij} C_2^{ij}$$

which is a constant independent of configuration ϖ . \square

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